ASEN5519 Topics in Multiphysics Modeling

External Structure-Acoustic Interactions

Basic Governing Equations for Acoustics

Helmholtz's Formula

Kirchhoff's Retarded Potential Equation and its Laplace Transform

Approximate Equations for Acoustic Field Problems

Coupled External Acoustic-Structure Interaction Equations

Basic Principles for Acoustics

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1}$$

Condensation, s**:**

$$\rho = \rho_o(1 + s) \tag{2}$$

Euler's equations:

$$\begin{aligned} &\rho \dot{\mathbf{v}} + (\rho \mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 \\ &\qquad \Downarrow \\ &\rho \dot{\mathbf{v}} = -c^2 \nabla \rho \qquad \Rightarrow \quad \dot{\mathbf{v}} = -c^2 \nabla s \\ &c^2 = (\frac{dp}{d\rho})_o \end{aligned} \tag{3}$$

where it is assumed that the pressure is a function of the density alone.

For sound waves of small amplitudes, from (1) and (2) we have

$$\nabla \cdot \mathbf{v} = -\dot{s} \tag{4}$$

Velocity potential, ϕ: The velocity potential is defined as

$$\mathbf{v} = -\nabla\phi \tag{5}$$

From (4) and (5) we obtain

$$\frac{ds}{dt} = \nabla^2\phi \tag{6}$$

From (3) and (5) we obtain

$$c^2 s = \frac{\partial\phi}{\partial t} \tag{7}$$

Finally, from (6) and (7) we obtain the wave equation in terms of the velocity potential, ϕ:

$$\boxed{\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}} \tag{8}$$

Wave equation in terms of pressure, p: Since we have

$$p = p_0 + (\frac{\partial p}{\partial\rho})_0\ \rho' + ... = p_0 + c^2\rho', \quad \rho = \rho_0 + \rho' \tag{9}$$

one has

$$\nabla^2 p = c^2 \nabla^2 \rho' \tag{10}$$

From (1) and thefirst of (3), one obtains

$$\begin{gathered}\frac{1}{c^2}\ddot{p} = \ddot{\rho} = -\nabla(\rho_0 \dot{\mathbf{v}}) = -\nabla(-\nabla p) = \nabla^2 p \\ \Downarrow \\ \boxed{\nabla^2 p - \frac{1}{c^2}\ddot{p} = 0}\end{gathered} \tag{11}$$

Wave equation in terms of the particle velocity: Differentiating the second of (3) once in time yields

$$\rho_0 \ddot{\mathbf{v}} = -c^2 \nabla \dot{\rho} \tag{12}$$

Substituting $\dot{\rho}$ from (1) results in

$$\begin{gathered}\rho_0 \ddot{\mathbf{v}} = -c^2 \nabla(-\nabla \cdot (\rho \mathbf{v})) \\ \Downarrow \\ \boxed{\frac{1}{\rho_0}\nabla(\nabla \cdot (\rho \mathbf{v})) - \frac{1}{c^2}\frac{\partial^2 \mathbf{v}}{\partial t^2} = 0}\end{gathered} \tag{13}$$

Isotropic Spherical Waves of Sound

For spherical coordinates, Poisson's operator takes the form of

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \psi^2} \tag{14}$$

where r is the radial coordinate, θ and ψ are the two spherical angles.

For the case of *isotropic* wave propagation, the velocity potential depends only on r, which reduces ∇^2 to

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \tag{15}$$

so the wave equation(8) becomes

$$\boxed{\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}}$$

$$\Downarrow \tag{16}$$

$$\boxed{\frac{\partial^2}{\partial r^2}(r\phi) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\phi)}$$ (This has the form of the equation for plane waves!)

whose solution is given by

$$\phi = \frac{1}{r}\left\{ f(ct - r) + F(ct + r) \right\} \tag{17}$$

Understanding Spherical Waves

- $\phi = \frac{1}{r}\, f(ct - r)$ is the velocity potential of isotropic spherical waves diverging from the origin of the spherical coordinates ($r = 0$). On the other hand, $\phi_1 = \frac{1}{r}\, F(ct + r)$ represents waves converging to the origin at which it becomes singular, thus not physically attainable.

- The radial wave velocity $v(r)$ is obtained via (5) as

$$v(r) = -\frac{\partial \phi}{\partial r} = \frac{1}{r^2} f(t - r/c) + \frac{1}{r}\frac{\partial}{\partial r} f(t - r/c) \tag{18}$$

  Observe that when $r << R$, where R is a characteristic length, the term, $\frac{1}{r^2} f(t - r/c)$ dominates. As time passes when $r >> R$, then the term, $\frac{1}{r}\frac{\partial}{\partial r} f(t - r/c)$ dominates. This means for expanding spherical waves, the partical velocity not only undergoes attenuation but also changes its dominant component from $f(ct - r)$ to $\frac{\partial}{\partial r} f(ct - r)$.

- *A single source:* The flow rate at r is given by

$$\dot{q} = 4\pi r^2 v(r) = 4\pi\, f(t - r/c) + 4\pi r \frac{\partial}{\partial r} f(t - r/c) \tag{19}$$

  Observe that the flow rate at the origin becomes

$$\lim_{r \to 0} \dot{q} \to 4\pi\, f(t) \tag{20}$$

which means that the expanding spherical waves are caused by *a simple source* and call $f(t)$ the strength of the simple source.

- *A double source:* Let us consider a single source of strength $f(t)$ at $P'(x', y,', z')$ and another of strength $-f(t)$ at (x'+ $\ell h, y' + mh, z' + nh)$ where (ℓ, m, n) are the direction cosines of the line joining the two sources. The velocity potential at (x, y, z) due to the source $f(t)$ at (x', y', z') may be expressed as

$$\phi = \frac{1}{r} f(t - r/c) = \phi(x - x', y - y', z - z', t) \tag{21}$$

Therefore, the velocity potential due to the two adjacent two sources can be expressed as

$$\begin{aligned} \phi &= \phi(x - x', y - y', z - z', t) - \phi(x - x' - \ell h, y - y' - mh, z - z' - nh, t) \\ &\approx (\ell \frac{\partial}{\partial x} + m \frac{\partial}{\partial x} + n \frac{\partial}{\partial x})\ h\phi(x - x', y - y', z - z', t) + O(h^2) \\ &= (\ell \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z})\ hf(t - r/c) \end{aligned} \tag{22}$$

If we assume

$$hf(t) = const = F(t) \tag{23}$$

then the velocity potential due to two adjacent sources (a double source) is given by

$$\lim_{h \to 0} \phi = (\ell \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z})\ \frac{1}{r} F(t - r/c) \tag{24}$$

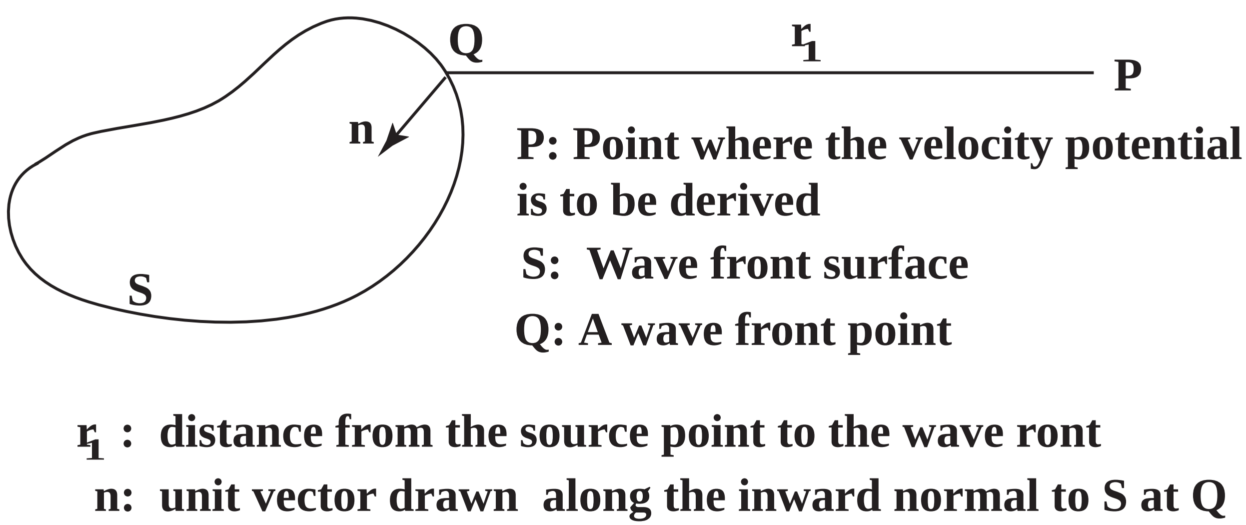
Helmholtz's Formula

There are two ways for deriving Helmholtz's formula: using Green's identity and employing physical considerations. We will employ the second approach.

First, it can be shown that the velocity potential given by

$$\phi = \bar{\phi} e^{ikct} \tag{25}$$

satisfies the wave equation (9). In equation (25), k is the monochromatic wave number, c is the speed of the wave, and $\phi(r)$ the strength of the potential.



Derivation of the velocity potential at P due to sources inside S

Let **n** is the normal unit vector drawn inward to S at Q. The particle velocity component v_n of in the outward direction of the normal vector, $\bar{\mathbf{n}} = -\mathbf{n}$, is thus obtained by

$$v_n = -\frac{\partial \phi}{\partial \bar{n}} = \frac{\partial \phi}{\partial n} = \frac{\partial \bar{\phi}}{\partial n} e^{ikct} \tag{26}$$

The flow rate (or flux), q, across the surface element dS at is given by

$$\dot{q} = v_n dS = \frac{\partial \bar{\phi}}{\partial n} e^{ikct} dS \tag{27}$$

In addition, the sources within S changes the pressure at Q from the equilibrium condition p to $p + dp$, from which one obtains from (2) and (7)

$$dp = (\frac{\partial p}{\partial \rho})_o \, d\rho = c^2 \rho_o s = \rho_o \frac{\partial \phi}{\partial t} = -i\rho_o kc\bar{\phi}(r) e^{ikct} \tag{28}$$

Therefore, the total thrust due to this pressure variation will be

$$dT = dp \, dS = -i\rho_o kc\bar{\phi}(r) e^{ikct} \, dS \tag{29}$$

It was Fresnel who observed that the flux, $\dot{q}$, being created at the surface element dS, can be considered as a source which gives rise to the velocity potential at P by

$$d\phi(P, t)_v = \frac{1}{4\pi r_1} \frac{\partial \bar{\phi}}{\partial n} e^{ik(r_1 - ct)} dS \tag{30}$$

it now remainsto consider the velocity potential due to the creation of the concentrated thrust dT at Q. This can be modeled by treating the pressure creation as a double source. Therefore, employing (24) one obtains the velocity potential due to the pressure change as

$$d\phi_p(P,t) = -\frac{\bar{\phi}}{4\pi}\frac{\partial}{\partial n}\left(\frac{e^{ik(r_1-ct)}}{r_1}\right)dS \tag{31}$$

Finally, the total velocity potential at P is given by

$$\phi(P,t) = \int_S \{d\phi(P,t)_v + d\phi(P,t)_p\}\, dS$$
$$\Downarrow \tag{32}$$
$$\boxed{\phi(P,t) = \frac{1}{4\pi}\int_S \{\frac{e^{ik(r_1-ct)}}{r_1}\frac{\partial\bar{\phi}}{\partial n} - \bar{\phi}\frac{\partial}{\partial n}(\frac{e^{ik(r_1-ct)}}{r_1})\}\, dS}$$

This is Helmholtz's formula, which is valid when P lies outside the closed surface S, and when P inside S if there is no source within it. If P is inside S and there are sources within it, we have $\phi(P,t) = 0$. (why?)

For the semi-infinite case where the domain is being bounded only by a plane, the constant, $(\frac{1}{4\pi})$,is modified to $(\frac{1}{2\pi})$. The reason is that (32) can be rearranged as

$$\lim_{r\to 0}\int \phi(P,t)\frac{\partial}{\partial n}(\frac{1}{r})dS' - \int_S \{\frac{e^{ik(r_1-ct)}}{r_1}\frac{\partial\bar{\phi}}{\partial n} - \bar{\phi}\frac{\partial}{\partial n}(\frac{e^{ik(r_1-ct)}}{r_1})\}\, dS = 0 \tag{33}$$

where dS' is the surface area of a sphere with its radius of r.

For an infinitely small radius of a sphere, we have

$$dS' = r^2 d\theta d\psi, \qquad \frac{\partial}{\partial n}(\frac{1}{r}) = -\frac{1}{r^2} \tag{34}$$

which leads to

$$\lim_{r\to 0} \int \phi(P,t)\ \frac{\partial}{\partial n}(\frac{1}{r}) dS' = -\int_{r\to 0} \phi(P,t)\ \frac{dS'}{r^2} \tag{35}$$

The term $\{\int \frac{dS'}{r^2}\}$ is known as the *solid angle* subtended by $\triangle S'$ in the literature and takes on the following values:

$$\lim_{r\to 0} \int \phi(P,t)\ \frac{dS'}{r^2} = \begin{cases} 4\pi\ \phi(P,t), & \text{for a sphere} \\ 2\pi\ \phi(P,t), & \text{for a plane} \end{cases} \tag{36}$$

We now generalize Helmholtz's formula:

$$\boxed{4\pi\epsilon\ \phi(P,t) = \int_S \{\frac{e^{ik(r_1 - ct)}}{r_1}\frac{\partial\bar{\phi}}{\partial n} - \bar{\phi}\frac{\partial}{\partial n}(\frac{e^{ik(r_1 - ct)}}{r_1})\}\ dS} \tag{37}$$

$$\epsilon = \begin{cases} 0, & \text{for spherical waves and when P is away from S} \\ \frac{1}{2}, & \text{for plane waves and/or when P is on S} \\ 1, & \text{for spherical waves and when P is inside S} \end{cases}$$

Kirchhoff's Retarded Potential Formula

If ϕ is a function of the coordinates (x, y, z) and time t of a variable point Q

$$\phi = \phi(x, y, z, t) \tag{38}$$

then the function at P with a distance r from Q may be expressed as

$$[\phi] = \phi(x, y, z, t - r/c) \tag{39}$$

which is called the retarded value of ϕ.

We now express Helmholtz's formula in terms of the retarded velocity potential.

$$\begin{aligned} 4\pi\epsilon\ \phi(P,t) &= \int_S \{\, [\phi]\, e^{-ikr} \frac{\partial}{\partial n}(\frac{e^{ikr}}{r}) - \frac{1}{r}[\frac{\partial \phi}{\partial n}]\,\}\, dS \\ &\Downarrow \\ 4\pi\epsilon\ \phi(P,t) &= \int_S \{\, [\phi]\, \frac{\partial r}{\partial n}(\frac{ik}{r} + \frac{d}{dr}(\frac{1}{r})) - \frac{1}{r}[\frac{\partial \phi}{\partial n}]\,\}\, dS \end{aligned} \tag{40}$$

Since

$$\frac{\partial \phi}{\partial t} = -ikc\phi \quad \Rightarrow \quad [\frac{\partial \phi}{\partial t}] = -ikc[\phi] \tag{41}$$

equation(40) becomes

$$\boxed{4\pi\epsilon\ \phi(P,t) = \int_S \{\, -[\phi]\, \frac{\partial r}{\partial n}\frac{1}{r^2} - \frac{1}{cr}\frac{\partial r}{\partial n}[\frac{\partial \phi}{\partial t}] - \frac{1}{r}[\frac{\partial \phi}{\partial n}]\,\}\, dS} \tag{42}$$

Laplace Transform of the Retarded Potential

The retarded potential equation derived in(42) is extremely difficult to evaluate in time domain. Although a limited solution exists for external wave problems such as spheres and infinite cylinder, it is not tractable for general structural shapes. Hence, various approximations have been proposed (see, e.g.,Geers[1-3] and Felippa[4-5]. The underlying observations for approximating the pressure field approximations have been to obtain to limiting temporal domain, transient early time approximations when the waves start to initiate propagation and late-time approximations after waves reach their steady motions. Mathematically, these approximations to a varying degree utilizes the following properties:

$$\begin{aligned} \text{Initial value theorem:} \quad \lim_{s\to\infty} sF(s) &= \lim_{t\to 0} f(t) \\ \text{Final value theorem:} \quad \lim_{s\to 0} sF(s) &= \lim_{t\to\infty} f(t) \end{aligned} \tag{43}$$

Before we transform the retarded velocity potential equation derived in equation(42), it is convenient to express the third term in terms of the normal velocity on the surface, viz.,

$$\boxed{4\pi\epsilon\,\phi(P,t) = \int_S \{ -[\phi]\,\frac{\partial r}{\partial n}\frac{1}{r^2} - \frac{1}{cr}\frac{\partial r}{\partial n}[\frac{\partial \phi}{\partial t}] + \frac{1}{r}[u_n]\,\}\,dS} \quad \text{since} \quad [u_n] = -[\frac{\partial \phi}{\partial n}] \tag{44}$$

With $\Phi = \int_0^\infty e^{-st}\phi(x,y,z,t)\,dt$ and $U_n(s) = \int_0^\infty e^{-st}u_n(x,y,z,t)\,dt$, the preceding equation is transformed into

$$\boxed{4\pi\epsilon\ \Phi(P,s) = \int_S \{\ -\Phi(s)\ \frac{\partial r}{\partial n}\frac{1}{r^2} - \frac{1}{cr}\frac{\partial r}{\partial n}\{s\Phi(s) - \phi(0)\} + \frac{1}{r}U_n(s)\ \}\ e^{-rs/c}\ dS} \tag{45}$$

In the above equation, the delayed exponential $e^{-sr/c}$ comes from the following Laplace transform property:

$$\begin{aligned}&\int_0^\infty e^{-st}[\phi]dt = \int_0^\infty e^{-st}\phi(x,y,z,t-r/c)dt = \int_0^\infty e^{-s(t'+r/s)}\phi(x,y,z,t')dt' \\ &= e^{-sr/s}\int_0^\infty e^{-st'}\phi(x,y,z,t')dt' = e^{-sr/s}\Phi(x,y,z,s) = e^{-sr/s}\Phi(s)\end{aligned} \tag{46}$$

A similar transformation can be carried out for $[u_n(t)]$.

The solution of the pressure distributions and the normal velocity on the structural surface employing either the time-domain retarded potential(44) or the Laplace-transformed equation(45) presents considerable challenge for general structural geometries. This has led many investigators to seek approximations of them. This is discussed below.

Approximations of the Laplace-Transformed Retarded Potential

A key step in approximating the retarded potential (44) or its Laplace-transformed counterpart(45) is to expand various terms in a rapidly converging series. Hence, it has been observed that during the initial stage of the wave propagation, the domain of influence may be characterized by

$$t << R/c \tag{47}$$

where R is the characteristic length of the structure. This means one may approximate by expanding the velocity potential and velocity terms in time series since the initial value theorem in the Laplace transform suggests that $(t \to 0) \equiv (s \to \infty)$.

The so-called late time response may be characterized as

$$t >> R/c \tag{48}$$

which corresponds to $(t \to \infty) \equiv (s \to 0)$ so that it is advantageous in expanding the terms in the Laplace variable, s.

A closer examination of the exponential term , $e^{-(rs/c)}$, reveals the following insight.

Initial wave transient period: When a source at point P starts to propagate, one may argue qualitatively $(t \to \text{small}) \equiv (s \to \text{large})$. However, the distance to which the wave propagate has propagated is also short, i.e., $r << R$. Therefore, if we restricts ourselves the initial wave propagation period to be

$$rs/c < 1 \tag{49}$$

a series expansion of $e^{-(rs/c)}$ will rapidly converge.

This restriction effectively allows us to by pass the time-domain approximation of the retarded potential for early time responses.

Late time wave propagation period: Conversely, we have ($t \to$ large) $\equiv$ ($s \to$ small), whose net effect provides the same convergent expansion parameter range shown in (48).

The preceding observations allow us to expand the delayed exponential $e^{-(rs/c)}$ over the response time of interest.

Before we carry out a series expansion of $e^{-(rs/c)}$, we modify it for improved convergence, more importantly for stability, by introducing the following modification:

$$\begin{aligned} e^{-\mu} &= e^{-(1+\alpha)\mu+\alpha\mu}, \quad \mu = rs/c \\ e^{-\mu} &= \frac{e^{\alpha\mu}}{e^{(1+\alpha)\mu}} \\ &\Downarrow \\ e^{-\mu} &= \frac{1+\alpha\mu+\frac{1}{2}\alpha^2\mu^2+...}{1+(1+\alpha)\mu+\frac{1}{2}(1+\alpha)^2\mu^2+...}, \quad \alpha \geq 0 \text{ for stability.} \end{aligned} \tag{50}$$

It should be mentioned that from a purely truncation error point of view, the most accurate approximate expansion is known as the (2,2)-P adé approximation:

$$e^{-\mu} = \frac{1-\mu/2+\mu^2/12}{1+\mu/2+\mu^2/12} \tag{51}$$

Unfortunately, this seemingly accurate expansion gives rise to unstable pressure equation. (why?)

Substituting (50) into the Laplace-transformed retarded potential equation (45), we obtain

$$
\begin{aligned}
&2\pi\{1+(1+\alpha)\frac{rs}{c}+\frac{(1+\alpha)^2}{2}(\frac{rs}{c})^2+...\}\Phi(P,s)= \\
&-\int_S dS\,\frac{\partial r}{\partial n}\frac{1}{r^2}\,\{1+\alpha\frac{rs}{c}+\frac{\alpha^2}{2}(\frac{rs}{c})^2+...\}\Phi(s) \\
&-\int_S dS\,\frac{1}{cr}\frac{\partial r}{\partial n}\,\{1+\alpha\frac{rs}{c}+\frac{\alpha^2}{2}(\frac{rs}{c})^2+...\}(s\Phi(s)-\phi(0)) \\
&+\int_S dS\,\frac{1}{r}\,\{1+\alpha\frac{rs}{c}+\frac{\alpha^2}{2}(\frac{rs}{c})^2+...\}U_n(s)
\end{aligned}
\tag{52}
$$

In deriving an approximate formula, we will not use the powers of the Laplace variable, s. This is because we only restricted $rs/c < 1$, not s to be small. Instead, we exploit the convergence of the surface integral values as our truncation criterion. Specifically, we will retain the kernels in the spatial surface integrals of powers of $(1/r^2, 1/r, 1)$ and discard terms containing $(r^n, n > 0)$ from numerical convergence considerations.

The resulting approximate expression is thus obtained as

$$
\begin{aligned}
2\pi\Phi(P,s) &+ \int_S dS\, \frac{\partial r}{\partial n}\{\frac{1}{r^2} + \frac{1}{r}\frac{\alpha s}{c} + \frac{\alpha^2}{2}(\frac{s}{c})^2\}\Phi(s) \\
&+ \int_S dS\, \frac{\partial r}{\partial n}\,\{\frac{1}{rc} + \frac{\alpha s}{c^2}\}(s\Phi(s) - \phi(0)) \\
&= \int_S dS\;\{\frac{1}{r} + \frac{\alpha s}{c}\}U_n(s)
\end{aligned} \tag{53}
$$

For clarity, the preceding approximate equation is rearranged as

$$
\begin{aligned}
&(\alpha + \frac{\alpha^2}{2})B\, s^2\Phi(s) + (1+\alpha)cB_1\, s\Phi(s) + c^2B_2\,\Phi(s) = \alpha cA\, sU_n(s) + c^2A_1U_n(s) \\
&B = \int_S dS\,\frac{\partial r}{\partial n}, \quad B_1 = \int_S dS\,\frac{1}{r}\frac{\partial r}{\partial n}, \quad B_2 = \int_S dS\,\frac{1}{r^2}\frac{\partial r}{\partial n} + 2\pi\;\delta(P' - P) \\
&A = \int_S dS\,, \quad A_1 = \int_S dS\,\frac{1}{r}
\end{aligned} \tag{54}
$$

Transforming back to time domain and neglecting the initial conditions, one obtains:

$$
\boxed{(\alpha + \frac{\alpha^2}{2})B\,\ddot{\phi}(t) + (1+\alpha)cB_1\,\dot{\phi}(t) + c^2B_2\,\phi(t) = \alpha cA\,\dot{u}_n(t) + c^2A_1u_n(t)} \tag{55}
$$

In order to express the preceding equation in terms of the pressure on the structural surface, we first time-differentiate and multiply by the density ρ:

$$(\alpha + \frac{\alpha^2}{2})B\ \rho\dddot{\phi}(t) + (1+\alpha)cB_1\ \rho\ddot{\phi}(t) + c^2 B_2\ \rho\dot{\phi}(t) = \alpha\rho c A\ \ddot{u}_n(t) + \rho c^2 A_1 \dot{u}_n(t) \tag{56}$$

Substituting in terms of the pressure given by

$$p(t) = \rho\dot{\phi}(t) \tag{57}$$

we finally obtain the surface pressure subject to the surface normal as velocity

$$(\alpha + \frac{\alpha^2}{2})B\ \ddot{p}(t) + (1+\alpha)cB_1\ \dot{p}(t) + c^2 B_2\ p(t) = \alpha\rho c A\ \ddot{u}_n(t) + \rho c^2 A_1 \dot{u}_n(t) \tag{58}$$

In order to conform to classical expression, we introduce

$$M_a = \rho A B_2^{-1} A_1,\ M_n = \rho A B_2^{-1} B_1,\ Q = \frac{\rho}{c} A B_2^{-1} A,\ Q_n = \frac{\rho}{c} A B_2^{-1} B \tag{59}$$

where M_a is known as *added mass* in the literature. With these symbols (58) is expressed as

$$\boxed{(\alpha + \frac{\alpha^2}{2})Q_n\ \ddot{p}(t) + (1+\alpha)M_n\ \dot{p}(t) + \rho c A p(t) = \alpha\rho c Q\ \ddot{u}_n(t) + \rho c M_a\ \dot{u}_n(t)} \tag{60}$$

Specialization of the Present Approximation (60)**: First-order pressure differential equation**

For $\alpha = 0$, equation(60) reduces to

$$\text{Present Approximation:} \quad M_n \, \dot{p}(t) + \rho c A p(t) = \rho c M_a \, \dot{u}_n(t) \tag{61}$$

The so-called DAA1 approximation proposed by Geers is given by

$$\text{Geers' DAA1:} \quad M_a \, \dot{p}(t) + \rho c A p(t) = \rho c M_a \, \dot{u}_n(t) \tag{62}$$

The boundary integrals used in the precedingtwo equations are given by

$$\begin{aligned} M_a &= [\int_S dS] \, [\int_S dS \, \frac{1}{r^2} \frac{\partial r}{\partial n} + 2\pi \, \delta(P' - P)]^{-1} \, [\int_S dS \, \frac{1}{r}] \\ M_n &= [\int_S dS] \, [\int_S dS \, \frac{1}{r^2} \frac{\partial r}{\partial n} + 2\pi \, \delta(P' - P)]^{-1} \, [\int_S dS \, \frac{1}{r} \frac{\partial r}{\partial n}] \end{aligned} \tag{63}$$

Observations:

1. A key difference between the added mass M_a and M_n (which will be denoted as a projected added mass herein) is the pointwise projection term, $\frac{\partial r}{\partial n}$, that forms the cosine angle between the radial vector **r** from the source point P and the normal **n** to the surface at P.

2. Since $|\frac{\partial r}{\partial n}| < 1$, the uncoupled characteristic time constant of the eigenvalues of the present homogeneous equation (61) of the present approximation would be higher than that of the Geers approximation(62). A full ramification of the difference of the two approximations remains to be carefully examined.

Specialization of the Present Approximation (60)**: Second-order pressure differential equation**

For $\alpha \neq 0$, The present approximation is recalled from (60) as

Present Approximation:

$$(\alpha + \frac{\alpha^2}{2}) Q_n \; \ddot{p}(t) + (1+\alpha) M_n \; \dot{p}(t) + \rho c A p(t) = \alpha \rho c Q \; \ddot{u}_n(t) + \rho c M_a \; \dot{u}_n(t) \tag{64}$$

where α is a Padé regularization parameter which should be chosen from numerical experiment for high accuracy.

The DAA2 by Geers can be expressed as

Geers' DAA2:

$$\frac{1}{\beta \rho c} M_a A^{-1} M_a \; \ddot{p}(t) + \frac{1}{\beta} M_a \; \dot{p}(t) + \rho c A p(t) = \frac{1}{\beta} M_a A^{-1} M_a \; \ddot{u}_n(t) + \rho c M_a \; \dot{u}_n(t) \tag{65}$$

where $0 \leq \beta \leq 1$ is a parameter to be determined as the ratio of the pressure time constant vs. the structural frequency. In practice, it is taken as unity.

Coupled External Acoustic-Structure Interaction Equations

So far we have not addressed the structural equations motion interacting with the external acoustic medium. For this case the external forces consist of two parts: the applied forces that act directly on the structures and the acoustic pressure being applied by the medium on the acoustic boundaries. Thus, the structural equations can be written as

$$M_s \ \ddot{\mathbf{x}} + D_s \ \dot{\mathbf{x}} + K_s \ \mathbf{x} = \mathbf{f}^{ext} - G^T A(\mathbf{p}_I + \mathbf{p}_s) \tag{66}$$

where G^T extracts the normal component of the pressure that acts on the interface boundaries.

Hence, the coupled equations of motion consist of the following equation set if $\alpha = 0$ is invokes:

$$\boxed{\begin{array}{l} M_n \ \dot{\mathbf{p}}_s(t) + \rho c A \mathbf{p}_s(t) = \rho c M_a \ \dot{\mathbf{u}}_n(t), \quad \mathbf{u}_n = \mathbf{n} \cdot \dot{\mathbf{x}}_s \\ M_s \ \ddot{\mathbf{x}} + D_s \ \dot{\mathbf{x}} + K_s \ \mathbf{x} = \mathbf{f}^{ext} - G^T A(\mathbf{p}_I + \mathbf{p}_s) \end{array}} \tag{67}$$

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