Computational Methods for Transient Coupled-Physics Problems
(KAIST Fall 2005/KCP)

18 October 2005: Basic Transient Analysis Techniques

20 October 2005: Staggered Solution Procedures

25 October 2005: Structure-Acoustic Interaction Analysis

27 October 2005: Stabilization Strategies and their Applications

01 November 2005: Fine Points in Transient Analysis Techniques

03 November 2005: Simulation of Multi-Physics Problems

Introductory Notes: In the previous lecture, we have utilized a localized version of Lagrange multipliers to account for interface coupling. For the modeling of structure-acoustic interaction problems, it is possible to extend the same coupling treatment if both the structure and acoustic medium are modeled by the finite element method or a conventional boundary element method. For example, the structure-internal acoustic interaction problems can be treated by one of the four computational methods discussed in the previous lecture.

In what follows we will employ a model equation for the acoustic field obtained by approximating the Kirchhoff retarded potential which inherently includes the interactions on the acoustic field boundary as well as the acoustic sources, which is known in the literature as *doubly asymptotic approximation*. Hence, the interactions effected by the structural boundaries on the acoustic field are incorporated. In this lecture we will use this model to examine computational methods for solving the structure-external acoustic interaction problems. Briefly speaking, this model accounts for an added mass effect on the structural boundary resulted from approximating the infinite domain to capture low-frequency components, the plane wave approximation to asymptotically model the high-frequency limits, and the wave front particle bombardments by the structural motion.

Approximate External Structure-Acoustic Interaction Equations which we are about to use for the study of computational methods are given by

$$\boxed{\begin{array}{l} M_a\, \dot{\mathbf{p}}_s(t) + \rho c A \mathbf{p}_s(t) = \rho c M_a\, \dot{u}_n(t), \quad u_n = \mathbf{n} \cdot \dot{\mathbf{x}}_s \\ M_s\, \ddot{\mathbf{x}} + D_s\, \dot{\mathbf{x}} + K_s\, \mathbf{x} = \mathbf{f}^{ext} - G^T A(\mathbf{p}_I + \mathbf{p}_s), \quad \dot{\mathbf{x}} = \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_I \end{array}} \tag{1}$$

where $\mathbf{p}_s$ is the scattering pressure acting on the surface of the structure, $\mathbf{p}_I$ is the incident pressure acting on the structural surface, $\mathbf{u}_n$ is the normal scattering particle velocity of the acoustic medium due to the structural surface velocity, $\mathbf{x}$ is the structural displacement, ρ is the density of the acoustic medium, c is the speed of the sound of the acoustic medium, $\mathbf{M}_n$ added mass of the acoustic medium mapped on the structural surface, A is the area of the structural surface, $\mathbf{G}^T$ is the Boolean surface degrees-of-freedom extraction operator from the structural velocity, $(\mathbf{M}_s, \mathbf{D}_s, \mathbf{K}_s)$ are the mass, damping and stiffness matrix of the structure, respectively.

Characterization of the interaction equations

In the case of structure-structure interactions or the interactions of one second-order with another second-order systems, we know the nature of their characteristic roots. As the present model equation set (1) consists of a second-order system interacting with a parabolic system, it will prove to be useful to understand their combined characteristics, especially in terms of the coupling *strength*. This can be studied by employing the following two-DOF model:

$$\begin{aligned} &\dot{y} + \mu y = \rho c \ddot{x}, \quad \mu = \rho c a / m_f \\ &\ddot{x} + \omega^2 x = -\frac{m\mu}{\rho c} y, \quad \omega^2 = k_s / m_s, \quad m = m_f / m_s \end{aligned} \tag{2}$$

where a is the modal area,

The characteristic equation for (2) can be shown to be

$$(s+\nu)(s^2+\omega^2)+m\mu s^2=0 \tag{3}$$

It is seen the *coupling strength* is dictated by the modal mass ratio m. One way to examine how the coupling strength may influence the characteristic roots, thus the computational stability, is first draw the root loci of the characteristic equation(3). Figures 1 and 2 illustrate the movements of characteristics roots as the coupling strength m varies.

In Fig.1, we have chosen $\{\mu = 1, \omega = 1\}$. As the coupling strength m increases, the two imaginary roots at A' and A move to the zero located at the origin of the root loci. Note the imaginary roots at A' and A' represent the uncoupled structural dynamics characteristics. The uncoupled root of the acoustic equation is located at B which moves along the negative real toward the infnity. Observe for this case, for an intermediate coupling strength, i.e., $m = 0.905$, the interaction of

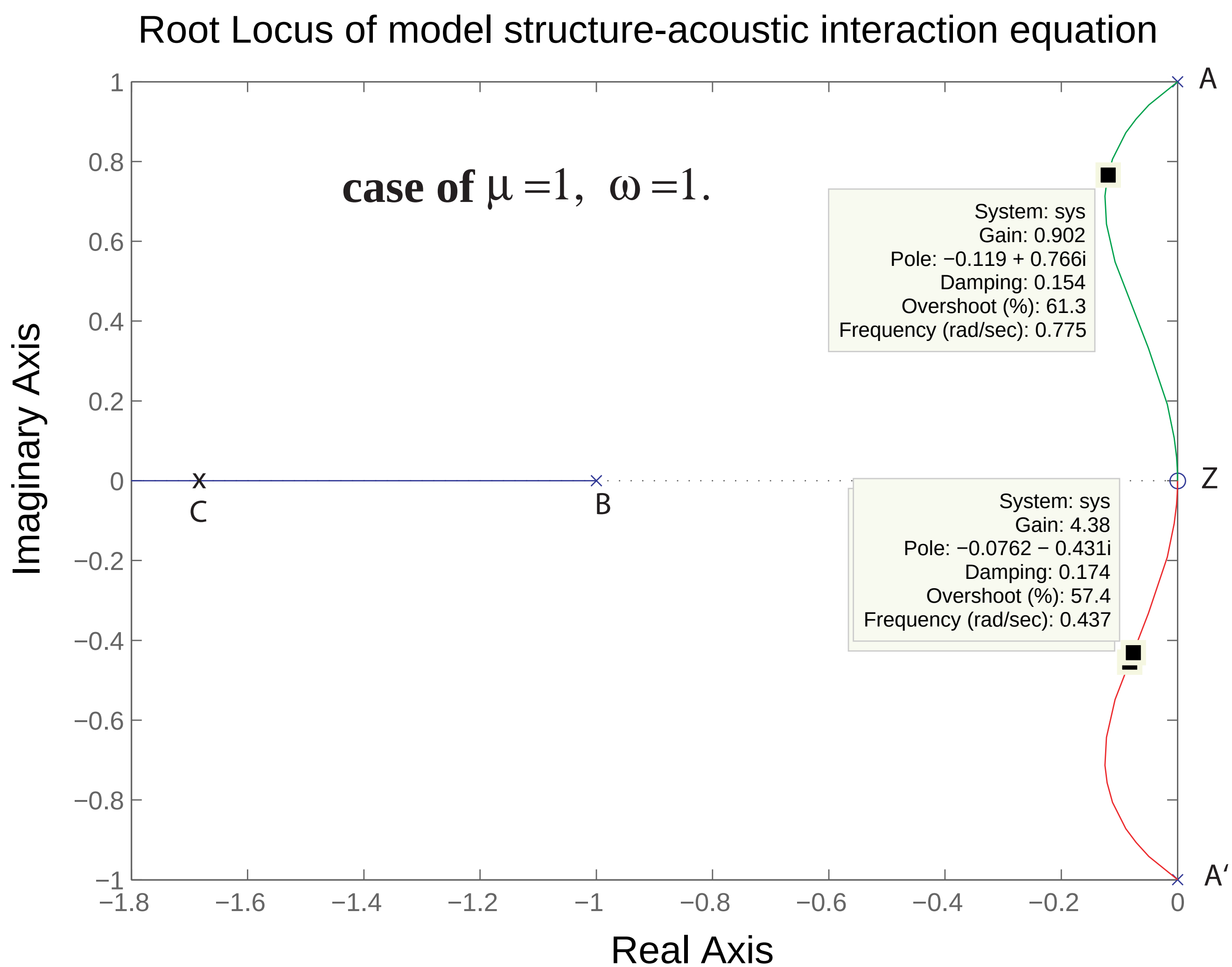

Fig. 1 Root loci of a coupled structure and external acoustic model

(Acoustic field interacting low frequency components of the structural system)

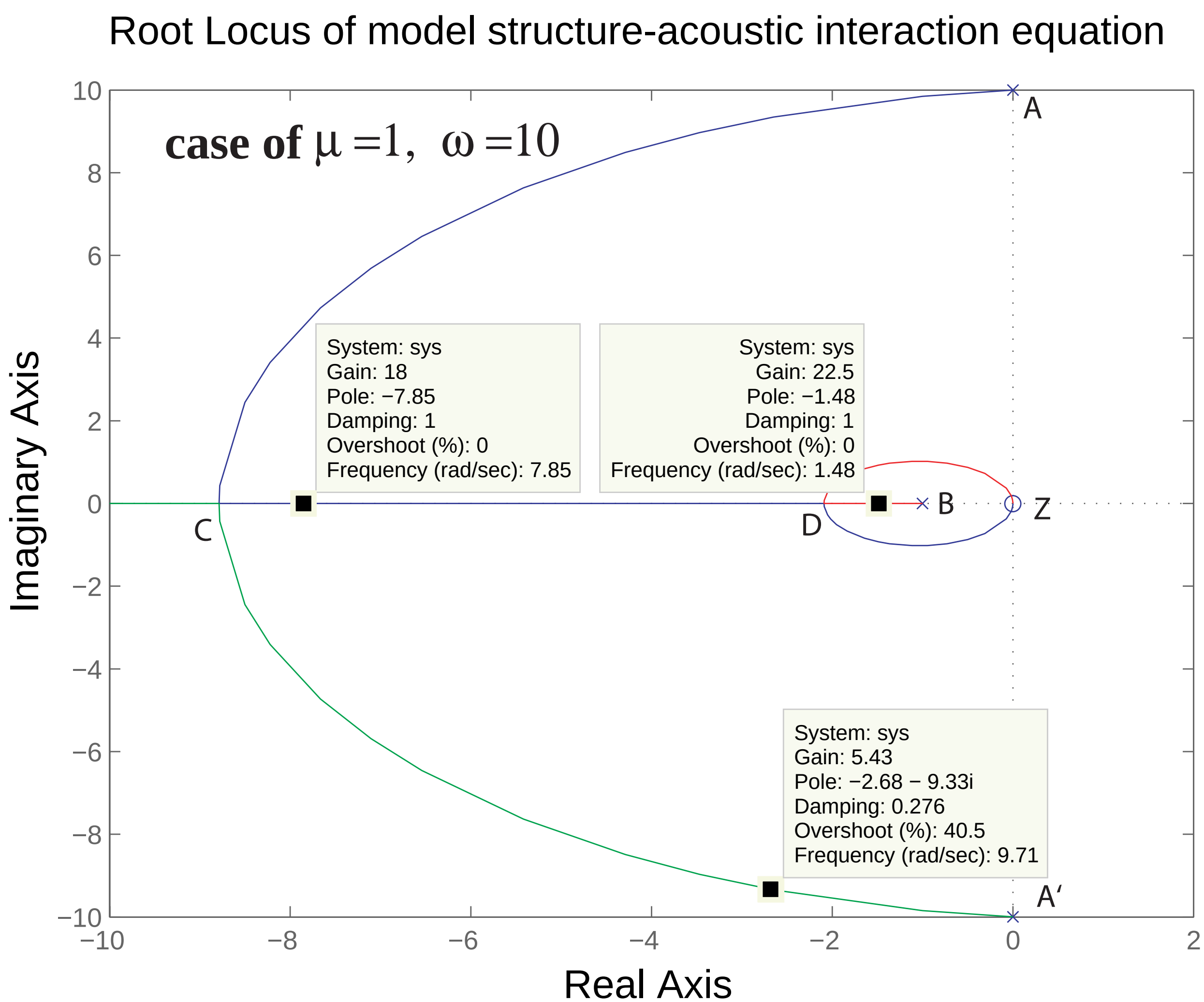

Fig. 2 Root loci of a coupled structure and external acoustic model

(Acoustic field interacting intermediate frequency components of the structural system) the structural system with the acoustic equation gives rise to a damped system. Physically this corresponds to the energy due to wave radiation into the infinite domain.

Figure 2 plots the root loci with the parameter set, $\{\mu = 1, \omega = 10\}$ which corresponds to the interaction of acoustic field with intermediate frequency components. Note that the structural roots follow a semi-circular path as the coupling strength increases. At point C, the loci form a double roots beyond which one root increases its magnitude to the negative axis. The uncoupled acoustic begins at B and moves toward to the negative axis. As the coupling strength further increases, the two roots meet at D. As the coupling strength is further increased, they form a conjugate pairs and approaches to the zero location for an infinitely large value of the coupling strength.

The above examination of the characteristics of the coupled system wil be utilized in the selection time integration algorithms and also stabilization.

Which Integration Formula(s) Should We Use?

It has been customary to pair the integration formulas with the problem types as follows:

Nonstiff parabolic problems: Runge-Kutter methods
Nonstiff hyperbolic problems: the Central difference formula
Stiff parabolic problems: Backward difference formulas
Stiff hyperbolic problems: the Trapezoidal rule or its variants

Here, *stiff problems* denote the discrete matrix systems whose largest eigenvalue is excessively large relative to its lowest one, or the condition number is very large. The coupled structure-external acoustic problem (1) is characterized as a combination of stiff hyperbolic system (structure) and stiff parabolic system(acoustic equation). Hence, we will employ a backward difference formula and the trapezoidal rule to time-discretize the coupled structure-external acoustic equation(1).

Specifically, we have chosen the following two-step backward difference formula:

$$y^{n+1} = \frac{4}{3}y^n - \frac{1}{3}y^{n-1} + \frac{2}{3}h\dot{y}^{n+1} \tag{4}$$

for the acoustic equation, and the trapezoidal rule for the structural equation:

$$\begin{aligned} x^{n+1} &= x^n + \frac{h}{2}(\dot{x}^{n+1} + \dot{x}^n) \\ \dot{x}^{n+1} &= \dot{x}^n + \frac{h}{2}(\ddot{x}^{n+1} + \ddot{x}^n) \end{aligned} \tag{5}$$

which is a full-step version of the trapezoidal rule, not the recommended form for actual implementation. However, as we are primarily focusing on computational stability, suffice it to use the full-step formula for this purpose.

Staggered Solution Procedure

In the previous lecture, we have studied various partitioned solution algorithms that exploits the use of single-field analysis software modules for the analysis of coupled field problems vis the method of localized Lagrange multipliers. The pivotal element that makes us to exploit the Lagrange multipliers is the availability of interface kinematic description, for example, the displacement at the interface to be the same. As stated in the beginning of this lecture, the present interaction formulation does not explicitly offer similar kinematic interface conditions as the interface effects directly act on the coupling terms, viz., the acoustic pressure acting on the structure and the structural particle velocity acting on the acoustic field. To this end, a method labeled as *staggered procedure* was developed to treat the type of coupled equations similar to the present case as shown in Fig. 3.

To advance from time $t = nh$ to the next step $t = (n+1)h$, we need to use an extrapolated value of y:

$$y_p^{n+1} = (1+\alpha)y^n - \alpha y^{n-1} \tag{6}$$

With this extrapolation, one can obtain the following difference equation set:

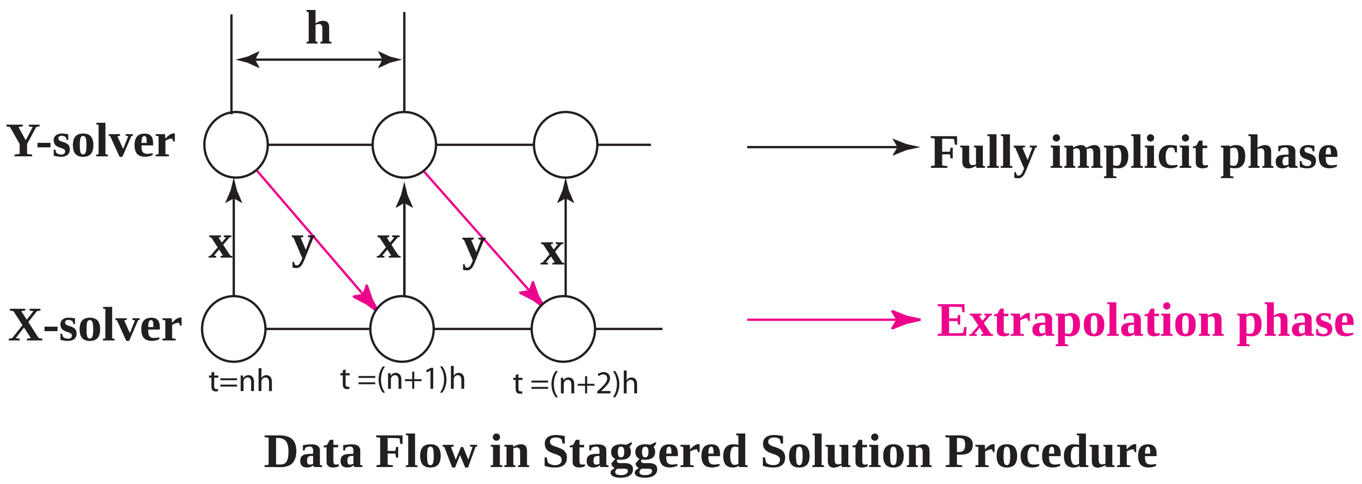

Fig. 3 Staggering Sequence for Solving Tightly Coupled Problems

$$\begin{aligned}
&3y^{n+1} - 4y^{n} + y^{n-1} + 2h\mu y^{n+1} - 2h\rho c\, \ddot{x}^{n+1} = 0 \\
&x^{n+1} - x^{n} - \frac{h}{2}(\dot{x}^{n+1} + \dot{x}^{n}) = 0 \\
&\dot{x}^{n+1} - \dot{x}^{n} - \frac{h}{2}(\ddot{x}^{n+1} + \ddot{x}^{n}) = 0 \\
&\ddot{x}^{n+1} + \omega^2 x^{n+1} + \frac{m\mu}{\rho c} y_p^{n+1} = 0
\end{aligned} \tag{7}$$

To assess the computational stability of the above difference equation set, we seek the solution of the form:

$$\begin{aligned} y^{n+1} &= \lambda y^n \\ x^{n+1} &= \lambda x^n \\ \dot{x}^{n+1} &= \lambda \dot{x}^n \\ \ddot{x}^{n+1} &= \lambda \ddot{x}^n \end{aligned} \tag{8}$$

Substituting this into the difference equation (7) yields

$$\begin{bmatrix} \hat{\rho}(\lambda) + h\mu\hat{\sigma}(\lambda) & 0 & 0 & -h\rho c\,\hat{\sigma}(\lambda) \\ 0 & \rho(\lambda) & h\sigma(\lambda) & 0 \\ 0 & 0 & \rho(\lambda) & h\sigma(\lambda) \\ \frac{m\mu}{\rho c}e(\lambda) & \lambda\omega^2 & 0 & \lambda \end{bmatrix} \begin{Bmatrix} y^n \\ x^n \\ \dot{x}^n \\ \ddot{x}^n \end{Bmatrix} = \mathbf{0} \tag{9}$$

where

$$\begin{aligned} &\hat{\rho}(\lambda) = (3\lambda^2 - 4\lambda + 1)/\lambda, \quad \hat{\sigma}(\lambda) = 2\lambda \\ &\rho(\lambda) = \lambda - 1, \quad \sigma(\lambda) = \tfrac{1}{2}(\lambda + 1), \quad e(\lambda) = (1+\alpha) - \frac{\alpha}{\lambda} \end{aligned} \tag{10}$$

The characteristic equation of the above difference equation set is found as

$$\lambda[\hat{\rho}(\lambda) + h\mu\hat{\sigma}(\lambda)][\rho^2(\lambda) + h^2\omega^2\sigma^2(\lambda)] + mh\mu\rho^2(\lambda)\hat{\sigma}(\lambda)e(\lambda) = 0 \tag{11}$$

We carry out a quantitative stability analysis via the Routh-Hurwitz criterion by transforming λ via

$$\lambda = \frac{1+z}{1-z} \tag{12}$$

to obtain the following z-polynomial equation:

$$[4(1+2z)z + 4h\mu(1+z)^2][(2z)^2 + h^2\omega^2] + mh\mu(2z)^2(1-z)[1+(1+2\alpha)z] = 0 \tag{13}$$

which can be rearranged as

$$
\begin{aligned}
&4[8+\mu h-(1+2\alpha)\mu hm]z^4+[16+8\mu h+8\alpha\mu hm]z^3\\
&+[\omega^2h^2(8+\mu h)+4\mu h(1+m)]z^2+2\omega^2h^2(2+\mu h)z\\
&+\omega^2h^2\mu h=0
\end{aligned}
\tag{14}
$$

Two extrapolations may be considered: $\alpha = 0$ and $\alpha = -\frac{1}{2}$:

$$
\begin{aligned}
\alpha = 0 \quad &\rightarrow \quad y_p^{n+1} = y^n\\
\alpha = -\tfrac{1}{2} \quad &\rightarrow \quad y_p^{n+1} = \tfrac{1}{2}(y^n + y^{n-1})
\end{aligned}
\tag{15}
$$

The Routh-Hurwitz stability test for the fourth-order characteristic polynomial

$$
f(z) = a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4, \quad a_0 > 0 \tag{16}
$$

can be examined carried out as follows.

First, we form the following Hurwitz matrix:

$$H = \begin{bmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{bmatrix} \tag{17}$$

Second, we construct the determinants of four submatrices

$$\begin{aligned} &\Delta_1 = a_1, \quad \Delta_2 = a_1 a_2 - a_0 a_3, \\ &\Delta_3 = a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4, \quad \Delta_4 = a_4 \Delta_3 \end{aligned} \tag{18}$$

Third, we require that the roots of $f(z) = 0$ will be in the right half-plane if and only if

$$\begin{aligned} &\{a_j > 0, \ j = 0, 1, 2, 3, 4\}, \\ &a_1 a_2 - a_0 a_3 > 0, \quad (a_1 a_2 - a_0 a_3) a_3 - a_1^2 a_4 > 0 \end{aligned} \tag{19}$$

Hence, the following conditions should meet:

For $\alpha = 0$:

$$8 + (1 - m)\mu h > 0, \quad m = m_f / m_s \tag{20}$$

For $\alpha = -\frac{1}{2}$:

$$2 + (1 - \tfrac{1}{2}m)h\mu > 0 \tag{21}$$

where m is the modal mass ratio of the acoustic medium to the structural mass.

A detailed analysis of the interaction model given by equation(1) as applied to a sphere surrounded an acoustic medium reveals that m can be greater than unity. Hence, we conclude that the staggered procedure outlined by equation set(7) is only conditionally stable. In fact, for all possible extrapolation of the pressure term acting on the structure leads to conditionally stable algorithm.

The above result of conditional computational stability of the structure-external acoustics problems when a staggered procedure is applied has led to a stabilization procedure which will be described in the next lecture.