# Partitioned Formulation of Structural-Acoustic Interaction Problems via Localized Lagrangian Multipliers

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## Abstract

A variational partitioned formulation of structural-acoustic interaction problems is presented by treating the interaction conditions via a localized version of the method of Lagrangian multipliers. The present formulation alleviates the impact of tightly coupled interaction models wherein the structural displacements on the interface act as the boundary condition for the fluid analyzer while the fluid pressure at the interaction boundary acts as the applied force for the structural analyzer. A general partitioned system interactions model is derived by employing a displacement model for the fluid and a localized method of Lagrangian multipliers. The resulting formulation conceptually separates the tight coupling of most existing formulations, thus offering software modularity in computer implementation. The general formulation is then specialized to a partitioned transient analysis procedure, a substructure-based vibration analysis model, and a fluid-pressure model suitable both for transient and vibration analyses.

## 1. Introduction

Formulation, discrete modeling and computer implementation of fluid-structure interaction problems have attracted many researchers in the past. As a result, a variety of fluid-structure interaction analysis approaches exist, ranging from invoking a tightly coupled integrated analysis procedure to using two independent fluid and structural analyzers in a staggered manner. For example, for vibration analysis of coupled fluid-structure interactions, it has been customary to use integrated equations of motion from which one may obtain the coupled vibratory modes and mode shapes (c.f., Ohayon, 1995). For the transient analysis of coupled fluid and structural problems, the fluid equations are time-integrated to offer the pressure at the wetted surfaces. This wetted surface pressure is then used as external forcing term to integrate the structural equations of motion. Once the structural velocity and displacement vectors are updated, they are then used as the boundary conditions for the fluid equations. This is designated as a staggered solution procedure (e.g., Park and Felippa, 1983).

Recently, multi-physics analysis that requires to couple more than two fields and parallel computations are emerging as a new challenge as well as new opportunities. These new challenges demand not only modularity of each single-field simulation capability, but also perhaps more importantly the interface phenomena are treated in separate module as much as possible. This means that a multi-physics oriented analysis of fluid-structure interaction problems may involve three modular attributes: a fluid analyzer, a structural analyzer, and an interface module that accounts for the fluid-structure interaction phenomena. This has motivated us to develop a formulation that leads to modular computational modeling and implementation of fluid-structure interaction problems.

In the context of variational formulations of the fluid-structure interactions, implementation modularity requires that the domain-to-domain interface mechanisms and field-to-field coupling phenomena must be modeled in terms of localized variables. For example, if the interface kinematical conditions are to be enforced in terms of the Lagrangian multipliers, the Lagrangian multipliers must possess a local attribute rather than global one. In a recent paper (Park and Felippa, 1998a) a variational formulation is presented, which treats the interface conditions for structure-structure partitions in terms of localized Lagrangian multipliers. As the localized treatment of Lagrangian multipliers constitutes a central role in the present formulation of fluid-structure interaction problems, we will first review the localized Lagrangian multipliers method using a discrete bar problem. A partitioned formulation of fluid-structure interaction problems will then follow.

## 2. Basic Concepts of Present Localized Lagrangian Multipliers: Discrete Bar Problem

Consider a two-bar system partitioned as shown in Fig. 1. For the assembled system shown by Fig. 1a, the system energy is given by

$$\begin{aligned} \Pi_g(\mathbf{u}_g) &= \mathbf{u}_g^T\ (\tfrac{1}{2}\mathbf{K}_g\mathbf{u}_g - \mathbf{M}_g\ddot{\mathbf{u}}_g - \mathbf{f}_g), \quad \mathbf{u}_g = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ \mathbf{K}_g &= \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad \mathbf{M}_g = \begin{bmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{f}_g = \begin{Bmatrix} 0 \\ P \end{Bmatrix} \end{aligned} \tag{1}$$

where $\mathbf{K}_g$ is the global or assembled stiffness matrix, $\mathbf{u}_g$ is the global nodal displacement, $\mathbf{f}_g$ is the force vector acting on the assembled system (a), and the subscript $(.)_g$ designates *global* quantities. The stationary value of the preceding functional, namely $\delta\Pi_g = 0$, yields the wellknown equilibrium equation:

$$\mathbf{M}_g\,\ddot{\mathbf{u}} + \mathbf{K}_g\,\mathbf{u}_g = \mathbf{f}_g \tag{2}$$

We now illustrate two distinct treatments of interface boundary conditions when the assembled structure is partitioned into two or more subdomains. Classical treatment of the partition boundaries by the method of Lagrangian multipliers is to enforce the interface kinematic conditions employing Newton's third law. For example, when a node in an assembled structure is partitioned into two substructural nodes, the partition boundary condition is realized by enforcing that the two partition boundary displacements to be the same. While the partitioned substructural displacements are 'localized', the Lagrangian mutiplier is of global nature in that it is common to the two partition boundary nodes.

In other words, the classical method of Lagrangian multipliers employs the 'localized' displacements in conjunction with 'globalized' multipliers in constructing the partition boundary constraint condition. It should be noted that the more the interface boundary condition is modeled in terms of localized variables, the more modular the resulting partitioned formulation will become.

In order to employ localized variables to a maximum degree in constructing the interface boundary constraint conditions, this paper introduces the following consideration. First, instead of requiring the partitioned substructural displacements to be the same, we propose that the two partitioned substructural boundary displacements to be equal to a global reference displacement. When this kinematical condition is enforced by Lagrangian multipliers, the corresponding Lagrangian multipliers become 'localized' variables. Consequently, the partition boundary condition is realized by 'local' substructural displacements, 'localized' Lagrangian multipliers, and a global reference displacement. Hence, the partition boundary condition is realized in terms of three variables, two localized and one global ones. We now illustrate the two methods via a simple example.

### 2.1 Classical Treatment of Partitioned Boundaries

Let us consider an assembled two-spring system partitioned into two springs as shown in Fig. 1b. When the classical Lagrange multipliers method is used to model the system, the energy functional consists of the energy of the two substructural springs which is augmented with the partition boundary constraint condition:

$$\Pi_c(\mathbf{u},\boldsymbol{\lambda}_{cl}) = \mathbf{u}^T\,(\tfrac{1}{2}\mathbf{K}\mathbf{u} - \mathbf{M}\ddot{\mathbf{u}} - \mathbf{f}) + \boldsymbol{\lambda}_{cl}^T\,\mathbf{C}_{cl}^T\,\mathbf{u}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & \\ 0 & 0 & m_2 \end{bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix}, \quad \boldsymbol{\lambda}_{cl} = \lambda_{12} \tag{3}$$

where the interface constraint is given by

$$c_{12} = u_1 - u_2 = \mathbf{C}_{cl}^T \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{C}_{cl}^T = \left[ \mathbf{C}_{cl}^{(1)^T} \quad \mathbf{C}_{cl}^{(2)^T} \right] = [\,[1] \quad [-1 \quad 0]\,] \tag{4}$$

in which $\mathbf{u}$ is the partitioned displacement, $\mathbf{K}$ is the partitioned stiffness matrix, $\mathbf{f}$ is the partitioned force vector acting on each spring, $\lambda_{12}$ is the interface reaction force to account for the constraint that the displacements $u_1$ and $u_2$ are the same, and the superscript $^{(j)}$ denotes springs 1 and 2. Note from Fig.1b that the Lagrange multiplier $\lambda_{12}$ is common to both the substructure (1) and (2), thus becoming a global variable.

The stationary condition $\delta \Pi_c(\mathbf{u}, \boldsymbol{\lambda}) = 0$ leads to the following partitioned equation set:

$$\mathbf{S} \left\{ \begin{array}{c} \mathbf{u} \\ \boldsymbol{\lambda} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{f} \\ \mathbf{0} \end{array} \right\}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{K}^{(1)} + \mathbf{M}^{(1)} \frac{d^2}{dt^2} & 0 & \mathbf{C}_{cl}^{(1)^T} \\ 0 & \mathbf{K}^{(2)} + \mathbf{M}^{(2)} \frac{d^2}{dt^2} & \mathbf{C}_{cl}^{(2)^T} \\ \mathbf{C}_{cl}^{(1)} & \mathbf{C}_{cl}^{(2)} & \mathbf{0} \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} k_1 + m_1 \frac{d^2}{dt^2} & 0 & 0 & 1 \\ 0 & k_2 + m_2 \frac{d^2}{dt^2} & -k_2 & -1 \\ 0 & -k_2 & k_2 + m_2 \frac{d^2}{dt^2} & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

Notice that the matrix differential operator $\mathbf{S}$ is indefinite. One solution method to bypass the indefiniteness is to eliminate the substructural displacements and solve for the Lagrangian multipliers. For example, implicit direct time integration of the preceding equation by the mid-point rule with its step size $\triangle t$:

$$\begin{aligned} \dot{\mathbf{u}}^{n+\frac{1}{2}} &= \dot{\mathbf{u}}^n + \frac{1}{\beta} \ddot{\mathbf{u}}^{n+\frac{1}{2}}, \quad \beta = \frac{2}{\Delta t} \\ \mathbf{u}^{n+\frac{1}{2}} &= \mathbf{u}^n + \frac{1}{\beta} \dot{\mathbf{u}}^{n+\frac{1}{2}}, \qquad \mathbf{u}^{n+1} = 2\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}^n \end{aligned} \tag{6}$$

leads to the following matrix discrete equation:

$$\bar{\mathbf{S}}_d \left\{ \begin{array}{c} \mathbf{u} \\ \boldsymbol{\lambda} \end{array} \right\}^{n+\frac{1}{2}} = \left\{ \begin{array}{c} \bar{\mathbf{f}} \\ \mathbf{0} \end{array} \right\}^{n+\frac{1}{2}}$$

$$\bar{\mathbf{S}}_d = \mathbf{S} = \begin{bmatrix} \mathbf{K}^{(1)} + \mathbf{M}^{(1)} \beta^2 & 0 & \mathbf{C}_{cl}^{(1)^T} \\ 0 & \mathbf{K}^{(2)} + \mathbf{M}^{(2)} \beta^2 & \mathbf{C}_{cl}^{(2)^T} \\ \mathbf{C}_{cl}^{(1)} & \mathbf{C}_{cl}^{(2)} & \mathbf{0} \end{bmatrix} \tag{7}$$

$$\bar{\mathbf{f}}^{n+\frac{1}{2}} = \mathbf{f}^{n+\frac{1}{2}} + \mathbf{M}(\beta^2 \mathbf{u}^n + \beta \dot{\mathbf{u}}^n)$$

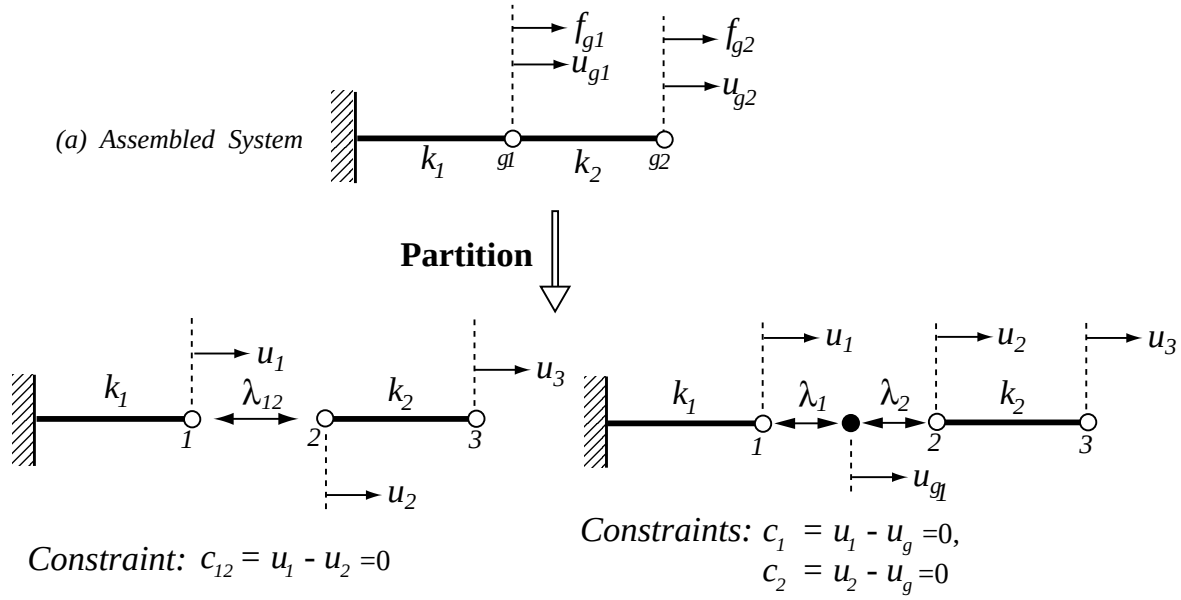


Figure 1. Partitioning of Two Springs
(a) Assembled system
(b) Partitioned system whose interface is modeled by classical $\boldsymbol{\lambda}$-method
(c) Partitioned system whose interface is modeled by localized $\boldsymbol{\lambda}$-method

Solving for the substructural displacements $\mathbf{u}$ for substructures (1) and (2), and substituting them into the interface compatibility condition, i.e., the last row of (7), we obtain

$$\{\mathbf{C}_{cl}^T\,[\mathbf{K}+\beta^2\mathbf{M}]^{-1}\,\mathbf{C}_{cl}\}\,\boldsymbol{\lambda}_{cl}^{n+\frac{1}{2}} = \mathbf{C}_{cl}^T\,[\mathbf{K}+\beta^2\mathbf{M}]^{-1}\,\bar{\mathbf{f}}^{n+\frac{1}{2}} \tag{8}$$

where $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}_{cl}$ are defined in (3) and (4). Specifically, the solution matrix for the Lagrangian multiplier $\boldsymbol{\lambda}_{12}$ becomes

$$\{\mathbf{C}_{cl}^T\,[\mathbf{K}+\beta^2\mathbf{M}]^{-1}\,\mathbf{C}_{cl}\} = [\frac{1}{(k_1+\beta^2 m_1)} + \frac{(k_2+\beta^2 m_2)}{D}], \quad D = \det|\mathbf{K}^{(2)}+\beta^2\mathbf{M}^{(2)}| \tag{9}$$

Observe that the solution matrix given by (9) is a dynamic flexibility matrix that is obtained by summing over the interface boundary attributes of both substructures (1) and (2). While the formation of the interface boundary flexibilities as a sum of all the interface substructural attributes is physically consistent, it may cause deleterious effects both on the conditioning of the solution matrix and on software modularity. For example, if the interface involves a solid and a plate, the resulting flexibility may experience high condition number that will adversely affect the solution accuracy as well efficiency if an iterative procedure is adotped. The loss of modularity, in addition to the non-unique issues associated with the classical $\boldsymbol{\lambda}$-method (Park and Felippa, 1998b), can

be a major concern for coupled-field analysis. This is addressed below by employing a localized version of the method of Lagrangian multipliers.

### 2.2 Localized Treatment of Partitioned Boundaries

Recently, a variational framework for treating the partition boundary constraints via a localized version of the method of Lagrangian multipliers was presented in Park and Felippa (1998a,1998b, 1998c). A distinctive feature of this localized treatment of interface constraints from the classical constraint condition (4) is that the partition boundary displacements are made equal to the global boundary displacement $\mathbf{u}_g$. Specifically, we have from Fig. 1c

$$\left\{\begin{array}{l} c_1 = u_1 - u_{g1} = 0 \\ c_2 = u_2 - u_{g1} = 0 \end{array}\right\} \quad \Rightarrow \quad \mathbf{B}_\ell^T(\mathbf{u} - \mathbf{L}\,\mathbf{u}_g) = 0 \quad \Rightarrow \quad \mathbf{B}_\ell^T\mathbf{u} - \mathbf{L}_b\,\mathbf{u}_b = 0$$
$$\mathbf{B}_\ell^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{L}_b = \mathbf{B}_\ell^T\,\mathbf{L} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{10}$$

where $\mathbf{B}_\ell$ is the partition-boundary extraction Boolean matrix, $\mathbf{L}$ is the well-known finite element assembly operator, and $\mathbf{u}_b$ is the global displacement pertaining to the partition boundary, respectively. For the example problem shown in Fig. 1c, we have $\mathbf{u}_b = u_{g1}$.

When (10) is augmented to the partitioned system energy (see Fig. 1c), the resulting modified partitioned system energy functional can be expressed by

$$\begin{aligned} \Pi_\ell(\mathbf{u}, \mathbf{u}_b, \boldsymbol{\lambda}_\ell) &= \mathbf{u}^T\,(\tfrac{1}{2}\mathbf{K}\mathbf{u} + \mathbf{M}\ddot{\mathbf{u}} - \mathbf{f}) + \boldsymbol{\lambda}_\ell^T(\mathbf{B}_\ell^T\mathbf{u} - \mathbf{L}_b\,\mathbf{u}_b) \\ \boldsymbol{\lambda}_\ell^T &= \langle\, \lambda_1 \quad \lambda_2 \,\rangle \end{aligned} \tag{11}$$

It is noted that the matrix $\mathbf{B}\ell$ is a localized Boolean matrix and the Lagrange multipliers $\boldsymbol{\lambda}_\ell^T = \langle\, \lambda_1 \quad \lambda_1 \,\rangle$ are also completely localized. The only global operator is $\mathbf{L}_b$-matrix that is global.

The constrained energy functional derived in (11) can be further decomposed by expressing the elemental displacements $\mathbf{u}$ into two parts(Park and Felippa, 1998a) :

$$\begin{aligned} &\mathbf{u} = \mathbf{d} + \mathbf{r}, \qquad \ddot{\mathbf{u}} = \ddot{\mathbf{d}} + \ddot{\mathbf{r}} \\ &\mathbf{d} = \boldsymbol{\Phi}\,\mathbf{q}, \qquad \mathbf{r} = \mathbf{R}\,\boldsymbol{\alpha}, \quad \mathbf{K}\,\mathbf{R} = 0 \end{aligned} \tag{12}$$

where $\mathbf{d}$ and $\mathbf{r}$ are the deformational and rigid-body displacements; $\boldsymbol{\Phi}$ and $\mathbf{R}$ are the deformation modes and rigid body modes; and, $\mathbf{q}$ and $\boldsymbol{\alpha}$ are the deformation and rigid-body amplitudes, respectively.

Substituting (12) into (11) leads to

$$\begin{aligned} \Pi(\boldsymbol{\lambda}_\ell, \boldsymbol{\alpha}, \mathbf{q}, \mathbf{u}_g) \quad = \quad & \mathbf{q}^T\boldsymbol{\Phi}^T\,[\tfrac{1}{2}\,\mathbf{K}\,\boldsymbol{\Phi}\,\mathbf{q} - \mathbf{f} + \mathbf{M}\,(\boldsymbol{\Phi}\,\ddot{\mathbf{q}} + \mathbf{R}\,\ddot{\boldsymbol{\alpha}})] \\ & + \boldsymbol{\lambda}_\ell^T\mathbf{B}_\ell^T\,(\boldsymbol{\Phi}\,\mathbf{q} - \mathbf{L}\,\mathbf{u}_g) \\ & + \boldsymbol{\alpha}^T\mathbf{R}^T\,[-\mathbf{f} + \mathbf{M}\,(\boldsymbol{\Phi}\,\ddot{\mathbf{q}} + \mathbf{R}\,\ddot{\boldsymbol{\alpha}}) + \mathbf{B}\,\boldsymbol{\lambda}_\ell] \end{aligned} \tag{13}$$

It should be noted that the four state variables ($\mathbf{q}$, $\boldsymbol{\lambda}_\ell$, $\boldsymbol{\alpha}$, $\mathbf{u}_g$) in the above equation are linearly independent provided that the constraint matrix $\mathbf{B}_\ell$ has full row rank. In other words, it is variationally complete. Thus, one may perform its variation to obtain

$$\begin{bmatrix} \boldsymbol{\Phi}^T\,(\mathbf{M}\dfrac{d^2}{dt^2}+\mathbf{K})\,\boldsymbol{\Phi} & \mathbf{0} & \boldsymbol{\Phi}^T\,\mathbf{B}_\ell & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^T\mathbf{M}\mathbf{R}\dfrac{d^2}{dt^2} & \mathbf{R}_b^T & \mathbf{0} \\ \mathbf{B}_\ell^T\,\boldsymbol{\Phi} & \mathbf{R}_b & \mathbf{0} & -\mathbf{L}_b \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_b^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \boldsymbol{\alpha} \\ \boldsymbol{\lambda}_b \\ \mathbf{u}_g \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\Phi}^T\,\mathbf{f} \\ \mathbf{R}^T\mathbf{f} \\ 0 \\ 0 \end{Bmatrix} \tag{14}$$

$$\mathbf{R}_b = \mathbf{B}_\ell^T\mathbf{R}, \quad \mathbf{L}_b = \mathbf{B}_\ell^T\mathbf{L}$$

In the preceding equation, the following identity

$$\begin{bmatrix} \boldsymbol{\Phi}^T \\ \mathbf{R}^T \end{bmatrix}^T \mathbf{M}\,[\,\boldsymbol{\Phi} \quad \mathbf{R}\,] = \mathbf{I} \tag{15}$$

is used to eliminate terms such as

$$\delta\mathbf{q}^T\boldsymbol{\Phi}^T\mathbf{M}\mathbf{R}\,\ddot{\boldsymbol{\alpha}} = 0, \quad \text{since} \quad \boldsymbol{\Phi}^T\mathbf{M}\mathbf{R} = \mathbf{0} \tag{16}$$

*Remark 2:* Comparing the two partitioned equations, viz., one augmented with the localized $\boldsymbol{\lambda}$-method (14) with the classical $\boldsymbol{\lambda}$-method (5), we observe the following:

- The boundary extraction operator $\mathbf{B}_\ell$ is completely localized. Hence, the present formulation does not require any summation of stiffness or other system-dependent attributes across the substructural interfaces.

- The kinematic interface compatibility condition given by the third row of (14)

$$\mathbf{B}_\ell^T\mathbf{u} - \mathbf{L}_b\mathbf{u}_b = \mathbf{B}_\ell^T\boldsymbol{\Phi}\mathbf{q} + \mathbf{R}_b\boldsymbol{\alpha} - \mathbf{L}_b\mathbf{u}_b = 0 \tag{14$c$}$$

  relates the partitioned interface displacement $\mathbf{u}$ to the assembled global displacement $\mathbf{u}_b$ at the partition boundary. This concept provides a unique way for constructing kinematic constraints when more than two nodes (or substructural boundaries) meet at the same discrete point.

- The fourth row of (14) given by

$$-\mathbf{L}^T\boldsymbol{\lambda}_b = 0 \quad \Rightarrow \quad -(\lambda_1+\lambda_2) = 0 \tag{14$c$}$$

  is the statement that the sum of reaction forces at a node disappears when the partitioned nodes are assembled together.

Specifically, the matrix components of the preceding partitioned equation for the example problem shown in Fig. 1c become:

$$\begin{aligned}
\mathbf{K} &= \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \\
\mathbf{B}_\ell &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
\mathbf{R}_b &= \mathbf{B}_\ell^T \mathbf{R} = \{\, 1 \,\}, \quad \mathbf{L}_b = \mathbf{B}_\ell^T \mathbf{L} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}
\end{aligned} \tag{17}$$

For an efficient solution of (14), we discretize in time the first row of (14) by the midpoint rule:

$$\begin{aligned}
\mathbf{q}^{n+\frac{1}{2}} &= \mathbf{q}^n + \frac{h}{2}\dot{\mathbf{q}}^{n+\frac{1}{2}} \\
\dot{\mathbf{q}}^{n+\frac{1}{2}} &= \dot{\mathbf{q}}^n + \frac{h}{2}\ddot{\mathbf{q}}^{n+\frac{1}{2}} \\
\mathbf{q}^{n+1} &= 2\mathbf{q}^{n+\frac{1}{2}} - \mathbf{q}^n
\end{aligned} \tag{18}$$

where $h$ is the stepsize, and the superscript $n$ designate the discrete time, $t = nh$.

Substituting the first two expressions of (18) into the first row of (12) leads to:

$$\begin{aligned}
&\bar{\mathbf{K}}_q\, \mathbf{q}^{n+\frac{1}{2}} = \mathbf{\Phi}^T\, (\mathbf{f}^{n+\frac{1}{2}} - \mathbf{B}\, \boldsymbol{\lambda}_b^{n+\frac{1}{2}}) + \mathbf{b}_q \\
&\bar{\mathbf{K}}_q = \mathbf{\Phi}^T\, (\mathbf{K} + \beta^2 \mathbf{M})\, \mathbf{\Phi}, \quad \beta = 2/h \\
&\mathbf{b}_q = \mathbf{\Phi}^T\, \{\mathbf{M}\, \mathbf{\Phi}\, (\beta^2 \mathbf{q}^n + \beta \dot{\mathbf{q}}^n)\,\} = \mathbf{\Phi}^T\, \mathbf{b}_u \\
&\mathbf{b}_u = \mathbf{M}\, (\beta^2 \mathbf{u}^n + \beta \dot{\mathbf{u}}^n)
\end{aligned} \tag{19}$$

where the last expression is obtained by using (12), (15) and (16).

It should noted that, as long as the mass matrix is positive definite, so is $\bar{\mathbf{K}}_q$. Hence, the preceding equation can be solved for the substructural displacement $\mathbf{q}^{n+\frac{1}{2}}$ by a suitable direct elimination technique, i.e., Cholesky method, to yield

$$\mathbf{q}^{n+\frac{1}{2}} = \bar{\mathbf{K}}_q^{-1}\, \{\, \mathbf{\Phi}^T\, (\mathbf{f}^{n+\frac{1}{2}} - \mathbf{B}\, \boldsymbol{\lambda}_b^{n+\frac{1}{2}}) + \mathbf{b}_q \} \tag{20}$$

Similarly, the second equation of (14) can be time-discretized as

$$\begin{aligned}
\bar{\mathbf{M}}_r \boldsymbol{\alpha}^{n+\frac{1}{2}} &= \mathbf{R}^T\, (\mathbf{f}^{n+\frac{1}{2}} - \mathbf{B}\boldsymbol{\lambda}_b^{n+\frac{1}{2}}) + \mathbf{b}_r \\
\bar{\mathbf{M}}_r &= \mathbf{R}^T (\beta \mathbf{C} + \beta^2 \mathbf{M}) \mathbf{R} \\
\mathbf{b}_r &= \mathbf{R}^T\, \{(\beta \mathbf{C} + \beta^2 \mathbf{M}) \mathbf{R} \boldsymbol{\alpha}^n + \beta\, \mathbf{M} \mathbf{R} \dot{\boldsymbol{\alpha}}^n\} = \mathbf{R}^T\, \mathbf{b}_u
\end{aligned} \tag{21}$$

Substituting $\mathbf{q}^{n+\frac{1}{2}}$ in (20) into the third row of (14) together with (21) and the last row of (14), we obtain the following equation for the interface force vector $\boldsymbol{\lambda}_\ell^{n+\frac{1}{2}}$:

$$\begin{bmatrix} \bar{\mathbf{F}}_b & -\mathbf{R}_b & \mathbf{L}_b \\ \mathbf{R}_b^T & \bar{\mathbf{M}}_r & \mathbf{0} \\ \mathbf{L}_b^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_\ell \\ \boldsymbol{\alpha} \\ \mathbf{u}_b \end{Bmatrix}^{n+\frac{1}{2}} = \begin{Bmatrix} \mathbf{g}_\lambda \\ \mathbf{g}_\alpha \\ \mathbf{0} \end{Bmatrix}$$
$$\mathbf{g}_\lambda = \mathbf{B}^T \left(\bar{\mathbf{K}}^{-1} - \mathbf{R}\, \bar{\mathbf{M}}_r^{-1}\, \mathbf{R}^T\right) \left(\mathbf{f}^{n+\frac{1}{2}} + \mathbf{b}_u\right) \tag{22}$$
$$\mathbf{g}_\alpha = \mathbf{R}^T \left(\mathbf{f}^{n+\frac{1}{2}} + \mathbf{b}_u\right)$$

where $\bar{\mathbf{F}}_b$ is given by

$$\bar{\mathbf{F}}_b = \mathbf{B}^T \boldsymbol{\Phi}\, \bar{\mathbf{K}}_q^{-1}\, \boldsymbol{\Phi}^T \mathbf{B} = \mathbf{B}^T \left(\bar{\mathbf{K}}^{-1} - \mathbf{R}\, \bar{\mathbf{M}}_r^{-1}\, \mathbf{R}^T\right) \mathbf{B},$$
$$\text{since} \quad \bar{\mathbf{K}} = [\mathbf{K} + \beta^2 \mathbf{M}]$$
$$= [\,\boldsymbol{\Phi} \quad \mathbf{R}\,] \begin{bmatrix} \Lambda_\phi + \beta^2 \mathbf{I} & 0 \\ 0 & \beta^2 \mathbf{I} \end{bmatrix} [\,\boldsymbol{\Phi} \quad \mathbf{R}\,]^T \tag{23}$$
$$\bar{\mathbf{K}}_q^{-1} = (\Lambda_\phi + \beta^2 \mathbf{I})^{-1}, \qquad \bar{\mathbf{M}}_r^{-1} = (\beta^2 \mathbf{I})^{-1}$$

The quastatic case of the preceding equation will be called *Partitioned Dynamic Flexibility Equation*. Note that even though it has been obtained by decomposing the substructural displacement $\mathbf{u}$ in terms of the deformation and rigid-body modes ($\boldsymbol{\Phi}\ \mathbf{R}$), the solution process does not require computations of the deformation modes $\boldsymbol{\Phi}$.

The preceding equation is derived for parallel computations in Park, Justino and Felippa (1997). An expository description of this equation viewed as a flexibility method is described in Felippa and Park (1997).

*Remark 3:* The preceding formalism possesses the following features:

- The procedure conceptually assumes the availability of the total assembled equilibrium equations;
- The partitioned equilibrium equation for each partition emanates by invoking the finite element *disassembly* process to the system energy functional;
- The resulting flexibility matrix $\bar{\mathbf{F}}_b$ forms a uncoupled block diagonal matrix:

$$\bar{\mathbf{F}}_b = \begin{bmatrix} \bar{\mathbf{F}}^{(1)} & . & \cdots & \cdots & \cdots \\ . & \bar{\mathbf{F}}^{(2)} & . & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \bar{\mathbf{F}}^{(n)} \end{bmatrix} \tag{24}$$

Specifically, for the example problem shown in Fig. 1c, it becomes a $(2\times 2)$ uncoupled matrix:

$$\bar{\mathbf{F}}_b = \begin{bmatrix} \frac{1}{(k_1+\beta^2 m_1)} & 0 \\ 0 & \frac{(k_2+\beta^2 m_2)}{D} - (2\beta^2 m_2)^{-1} \end{bmatrix}, \quad D = \det|\mathbf{K}^{(2)} + \beta^2 \mathbf{M}^{(2)}| \tag{25}$$

- The resulting Lagrangian multipliers possess a localized characteristic. It is this feature that will be exploited in the derivation of fluid-structure interaction problems in order to effect a modular implementation of interaction phenomena.

## 3. Variational Formulation of Elasto-Acoustic Problem by Primitive Variables

**3.1 Motivation:** One of the most widely adopted formulation of vibro-acoustic problems is a so-called unsymmetric $(u,\, p)$-formulation by Morand and Ohayon (1995). In order to achieve a symmetric formulation, they introduced a pressure potential with a proviso that the associated non-physical zero-mode is projected out a-priori. The resulting formulation, while achieving symmetry, introduces a highly dense mass matrix (For more details, see Section 8.4 of Morand and Ohayon). A closer examination of their formulation reveals that the pointwise enforcement of the interface condition between the fluid and structural surface dictated by

$$\sigma_{ij}\, n_j^S = -p\, n_i^S = p\, n_i \tag{26}$$

is responsible for subsequent tight coupling between the fluid and structural governing equations. In (26), $\sigma_{ij}$ is the stress tensor in the structure, $p$ is the fluid pressure, and $n$ is the unit normal to the interface surface.

In the present formulation to be described below, we bypass the above pointwise Neumann boundary condition by treating both the fluid and structure domains as a single continuum. A justification for this integrated continuum modeling of fluid and structure is that we substitute the pointwise interface condition by its weak constraint, viz., the sum of the discrete nodal forces to vanish at the fluid-structure interface. Specifically, we replace the point-wise interface normal stress condition (26) by a set of localized Lagrange multipliers and the associated kinematical relation

$$\mathcal{F}_f(\boldsymbol{\lambda}_f) + \mathcal{F}_s(\boldsymbol{\lambda}_s) = 0 \quad \text{and} \quad \left\{ \begin{matrix} \mathbf{u}_f \\ \mathbf{u}_s \end{matrix} \right\} = \left\{ \begin{matrix} \mathcal{F}_f^T \\ \mathcal{F}_s^T \end{matrix} \right\} \mathbf{u}_g \tag{27}$$

where $\boldsymbol{\lambda}_f$ and $\boldsymbol{\lambda}_s$ are Lagrange multipliers representing a set of generalized interacting forces on the fluid and structural interfaces; and, $\mathcal{F}_f$ and $\mathcal{F}_s$ are the mapping operators (or filters) that interpolate the reference interface displacements $\mathbf{u}_g$ on the fluid and structural interface displacements, $\mathbf{u}_f$ and $\mathbf{u}_s$, respectively.

Observe that $\boldsymbol{\lambda}_f$ and $\boldsymbol{\lambda}_s$ are independent, which offers the possibility of modeling them to reflect the predominant physics of each field. For example, the structure is in general characterized by much higher frequencies than the fluid. These distinct frequency characteristics can be exploited in approximating the two Lagrangian multipliers. This will be utilized in the construction of reduced models later in the present paper.

Figure 2. Fluid-Structure Interface Descriptions

**3.2 Partitioned Variational Formalism:** The variational formulation of a small-amplitude internal fluid vibro-acoustic problem in terms of displacement variables can be expresses as

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\Omega_s} \boldsymbol{\sigma}^T\ \boldsymbol{\epsilon}\ d\Omega_s + \tfrac{1}{2}\rho_f c_f^2 \int_{\Omega_f} (\nabla \mathbf{u}_f)^2 d\Omega_f \\
&+ \int_{\Omega_s} \mathbf{u}_s^T (\mathbf{f}_s - \rho_s \ddot{\mathbf{u}}_s)\ d\Omega_s + \int_{\Omega_f} \mathbf{u}_f^T (\mathbf{f}_f - \rho_f \ddot{\mathbf{u}}_f)\ d\Omega_f \\
&= \int_{\Gamma_s} \mathbf{u}_s^T \mathbf{T}_s\ d\Gamma + \int_{\Gamma_f} \mathbf{u}_f^T \mathbf{T}_f\ d\Gamma \\
&p = -\rho_f c_f^2\ \nabla \mathbf{u}_f \\
&\text{With condition: } \mathbf{curl}\ \mathbf{u}_f = 0 \quad \text{in} \quad \Omega_f
\end{aligned}
\tag{28}
$$

where $(\rho_s, \rho_f)$ are the density of the structure and fluid; $c_f$ is the sound speed of the fluid; $(\mathbf{u}_s, \mathbf{u}_f)$ are the structural and fluid displacement vectors; $(\boldsymbol{\sigma}, \boldsymbol{\epsilon})$ are the stress and strain vectors of the structure; $(\mathbf{f}_s, \mathbf{f}_f)$ are the body forces of the structure and fluid; and $(\Omega_s, \Omega_f)$ designate the interior structure and fluid domains, $(\Gamma_s, \Gamma_f)$ represent the physical boundaries of the structure and fluid; and, the superscript dots $\ddot{()}$ designate time differentiation.

The finite element discretization of (28) (see, for example, Section 8.2-4 of Morand and Ohayon) can be expressed as:

$$
\boxed{
\begin{aligned}
&\delta\Pi(\mathbf{u}_g) = \delta\mathbf{u}_g^T\ (\mathbf{K}_g\ \mathbf{u}_g + \mathbf{M}_g\ \ddot{\mathbf{u}}_g - \mathbf{F}_g) \\
&\mathbf{curl}\ \mathbf{u}_f \approx \mathbf{C}_f^T\ \mathbf{u}_f = 0 \\
&\mathbf{K}_g = [\,\mathbf{L}_s^T \quad \mathbf{L}_f^T\,] \begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_f \end{bmatrix} \begin{bmatrix} \mathbf{L}_s \\ \mathbf{L}_f \end{bmatrix} \\
&\mathbf{M}_g = [\,\mathbf{L}_s^T \quad \mathbf{L}_f\,] \begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_f \end{bmatrix} \begin{bmatrix} \mathbf{L}_s^T \\ \mathbf{L}_f \end{bmatrix} \\
&\begin{Bmatrix} \mathbf{u}_s \\ \mathbf{u}_f \end{Bmatrix} = \begin{bmatrix} \mathbf{L}_s \\ \mathbf{L}_f \end{bmatrix} \mathbf{u}_g \\
&\mathbf{f}_g = [\,\mathbf{L}_s^T \quad \mathbf{L}_f^T\,] \begin{Bmatrix} \mathbf{f}_{\gamma s} + \mathbf{t}_s \\ \mathbf{f}_{\gamma f} + \mathbf{t}_f \end{Bmatrix}
\end{aligned}
}
\tag{29}
$$

where $\mathbf{L}$ denotes a Boolean matrix that assembles elements to a global matrix, and various matrix and vectorial quantities are given by

$$
\begin{aligned}
&\delta \int_{\Omega_s} \boldsymbol{\sigma}^T \, \boldsymbol{\epsilon} \, d\Omega \approx \delta \mathbf{u}_s^T \, \mathbf{K}_s \, \mathbf{u}_s \\
&\rho_f \, c_f^2 \, \delta \int_{\Omega_f} (\nabla \mathbf{u}_f)^2 d\Omega \approx \delta \mathbf{u}_f^T \, \mathbf{K}_f \, \mathbf{u}_f \\
&\int_{\Omega_s} \rho_s \, \delta \mathbf{u}_s^T \ddot{\mathbf{u}}_s \, d\Omega \approx \delta \mathbf{u}_s^T \, \mathbf{M}_s \, \ddot{\mathbf{u}}_s \\
&\int_{\Omega_f} \rho_f \, \delta \mathbf{u}_f^T \ddot{\mathbf{u}}_f \, d\Omega \approx \delta \mathbf{u}_f^T \, \mathbf{M}_f \, \ddot{\mathbf{u}}_f \\
&\delta \int_{\Omega_s} \mathbf{u}_s^T \mathbf{f}_s \, d\Omega \approx \delta \mathbf{u}_s^T \, \mathbf{f}_{\gamma s} \\
&\delta \int_{\Omega_f} \mathbf{u}_f^T \mathbf{f}_f \, d\Omega \approx \delta \mathbf{u}_f^T \, \mathbf{f}_{\gamma f} \\
&\delta \int_{\Gamma_s} \mathbf{u}_s^T \mathbf{T}_s \, d\Gamma \approx \delta \mathbf{u}_s^T \, \mathbf{t}_s \\
&\delta \int_{\Gamma_f} \mathbf{u}_f^T \mathbf{T}_f \, d\Gamma \approx \delta \mathbf{u}_f^T \, \mathbf{t}_f
\end{aligned}
\tag{30}
$$

It should be noted that several investigators (Hamdi, Ousset and Verchery, 1978; Chen and Taylor, 1990; Bermudez and Rodriguez, 1994; Nédélec, 1979; Piet-Lahanier and Ohayon, 1990) proposed a similar formulation.

If we augment $\delta\Pi(\mathbf{u}_g)$ with the irrotational condition, we have the following modified functional

$$
\delta\Pi(\mathbf{u}_g, \boldsymbol{\mu}) = \delta \mathbf{u}_g^T \, (\mathbf{K}_g \, \mathbf{u}_g + \mathbf{M}_g \, \ddot{\mathbf{u}}_g - \mathbf{f}_g) + \delta \boldsymbol{\mu}^T \, \mathbf{C}_f^T \, \mathbf{u}_f + \delta \mathbf{u}_f^T \, \mathbf{C}_f \, \boldsymbol{\mu} \tag{31}
$$

Hence, the governing discrete equations of motion subject to the irrotational constraint leads to the following equation:

$$
\begin{bmatrix} \mathbf{K}_g + \mathbf{M}_g \, \frac{d^2}{dt^2} & \mathbf{C}_g \\ \mathbf{C}_g^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_g \\ \boldsymbol{\mu} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_g \\ \mathbf{0} \end{Bmatrix}, \qquad \mathbf{C}_g^T = [\, \mathbf{0} \quad \mathbf{C}_f^T \,] \begin{bmatrix} \mathbf{L}_s \\ \mathbf{L}_f \end{bmatrix} \tag{32}
$$

It should noted that for each fluid element, $\mathbf{C}$ becomes a $(3 \times n_f)$ where $n_f$ is the total elemental degrees of freedom. Care must be exercised in constructing the discrete irrotational operator in an analogous manner encountered in the incompressibility condition, which has been extensively studied (reference!) Alternatively, one can employ the fluid displacement interpolation basis that satisfies the irrotational condition a-priori (Raviart and Thomas, 1982).

## 4. Localized Partitioning of Fluid-Structure Interaction Equations

A key aspect in the localized treatment of partitioned-boundary condition as discussed in Sections 2.2 and 3.1 is to relate the partitioned-boundary displacements to the assembled displacement as

formulated in (14). When the partition boundary constitutes fluid-structure interface, this can be expressed as

$$\begin{aligned} \left\{ \begin{array}{c} \mathbf{u}_{fb} \\ \mathbf{u}_{sb} \end{array} \right\} - \mathcal{L}_b \, \mathbf{u}_b = 0 \quad \Rightarrow \quad \mathbf{B}_\ell^T \mathbf{u} - \mathcal{L}_b \, \mathbf{u}_b = 0, \quad \mathbf{u} = \left\{ \begin{array}{c} \mathbf{u}_s \\ \mathbf{u}_f \end{array} \right\} \\ \mathcal{L}_b = \mathcal{F} \, \mathbf{B}_\ell^T \mathbf{L} \, = \left[ \begin{array}{c} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{array} \right], \quad \mathcal{F} \, = \left[ \begin{array}{cc} \mathcal{F}_f & 0 \\ 0 & \mathcal{F}_s \end{array} \right], \quad \mathbf{B}_\ell = \left[ \begin{array}{cc} \mathbf{B}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f \end{array} \right], \end{aligned} \tag{33}$$

Hence, if the fluid domain is partitioned from the structural domain as shown in Fig. 2, the corresponding energy functional becomes

$$\begin{aligned} \delta \Pi(\mathbf{u}, \boldsymbol{\lambda}_\ell, \mathbf{u}_b, \boldsymbol{\mu}) = \delta \mathbf{u}^T \, (\mathbf{K} \, \mathbf{u} + \mathbf{M} \, \ddot{\mathbf{u}} - \mathbf{F}) + \delta \boldsymbol{\mu}^T \, \mathbf{C}_f^T \, \mathbf{u}_f + \delta \mathbf{u}_f^T \, \mathbf{C}_f^T \, \boldsymbol{\mu} \\ + \delta \boldsymbol{\lambda}_b^T \, (\mathbf{B}^T \mathbf{u} - \mathcal{L}_b \, \mathbf{u}_g) + (\delta \mathbf{u}^T \mathbf{B}_\ell - \delta \mathbf{u}_b^T \mathcal{L}_b^T \,) \, \boldsymbol{\lambda}_\ell \end{aligned} \tag{34}$$

which, when expanded into six-variable functional, becomes:

$$\begin{aligned} \delta \Pi(\mathbf{u}_s, \mathbf{u}_f, \boldsymbol{\lambda}_s, \boldsymbol{\lambda}_f, \mathbf{u}_b, \boldsymbol{\mu}) = \delta \mathbf{u}_s^T \, (\mathbf{K}_s \, \mathbf{u}_s + \mathbf{M}_s \, \ddot{\mathbf{u}}_s - \mathbf{f}_s + \mathbf{B}_s \, \boldsymbol{\lambda}_s) + \delta \boldsymbol{\mu}^T \, \mathbf{C}_f^T \, \mathbf{u}_f \\ + \delta \mathbf{u}_f^T \, (\mathbf{K}_f \, \mathbf{u}_f + \mathbf{M}_f \, \ddot{\mathbf{u}}_f - \mathbf{f}_f + \mathbf{C}_f \, \boldsymbol{\mu} + \mathbf{B}_f \, \boldsymbol{\lambda}_f) \\ + \delta \boldsymbol{\lambda}_\ell^T \, (\left[ \begin{array}{cc} \mathbf{B}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f^T \end{array} \right] \left\{ \begin{array}{c} \mathbf{u}_s \\ \mathbf{u}_f \end{array} \right\} - \left[ \begin{array}{c} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{array} \right] \mathbf{u}_b) \\ - \delta \mathbf{u}_b^T \, [ \, \mathcal{L}_{bs}^T \quad \mathcal{L}_{bf}^T \, ] \left\{ \begin{array}{c} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{array} \right\}, \quad \boldsymbol{\lambda}_\ell = \left\{ \begin{array}{c} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{array} \right\} \end{aligned} \tag{35}$$

It should be emphasized that the Lagrangian multipliers $\boldsymbol{\lambda}_\ell$ at the fluid interface nodes and the structure interface nodes are independently defined. Setting the variational expression $\delta \Pi(\mathbf{u}_s, \mathbf{u}_f, \boldsymbol{\lambda}_s, \boldsymbol{\lambda}_f, \mathbf{u}_b, \boldsymbol{\mu}) = 0$ we obtain the following partitioned equations of motion:

$$\left[ \begin{array}{cccccc} \mathbf{K}_s + \frac{d^2}{dt^2} \mathbf{M}_s & 0 & \mathbf{B}_s & 0 & 0 & 0 \\ 0 & \mathbf{K}_f + \frac{d^2}{dt^2} \mathbf{M}_f & 0 & \mathbf{B}_f & \mathbf{C}_f & 0 \\ \mathbf{B}_s^T & 0 & 0 & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \mathbf{B}_f^T & 0 & 0 & 0 & -\mathcal{L}_{bf} \\ 0 & \mathbf{C}_f^T & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{L}_{bs}^T & -\mathcal{L}_{bs}^T & 0 & 0 \end{array} \right] \left\{ \begin{array}{c} \mathbf{u}_s \\ \mathbf{u}_f \\ \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \boldsymbol{\mu} \\ \mathbf{u}_b \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{f}_s \\ \mathbf{f}_f \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} \tag{36}$$

This partitioned matrix equation for vibro-acoustic interaction problems is perhaps most general in that several special formulations can be derived from it. Three specializations will be subsequently discussed.

## 5. Partitioned Transient Analysis of Vibro-Acoustic Interactions

Time discretization of the partitioned vibro-acoustic interaction equation (36) can be carried out by integrating the structural and fluid displacement equations. To this end, we employ the mid-point integration rule given by

$$\begin{aligned}
\mathbf{u}^{n+\frac{1}{2}} &= \mathbf{u}^n + \tfrac{1}{2}\triangle t\dot{\mathbf{u}}^n + (\tfrac{1}{2}\triangle t)^2\,\ddot{\mathbf{u}}^{n+\frac{1}{2}} \\
\dot{\mathbf{u}}^{n+\frac{1}{2}} &= \dot{\mathbf{u}}^n + \tfrac{1}{2}\triangle t\,\ddot{\mathbf{u}}^{n+\frac{1}{2}} \\
\mathbf{u}^{n+1} &= 2\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}^n
\end{aligned} \tag{37}$$

where $\triangle t$ is the timestep size.

Integrating (33$a$) and (33$b$) by (37) yields

$$\begin{bmatrix}
\mathbf{K}_s + \beta^2\mathbf{M}_s & 0 & \mathbf{B}_s & 0 & 0 & 0 \\
0 & \mathbf{K}_f + \beta^2\mathbf{M}_f & 0 & \mathbf{B}_f & \mathbf{C}_f & 0 \\
\mathbf{B}_s^T & 0 & 0 & 0 & 0 & -\mathcal{L}_{bs} \\
0 & \mathbf{B}_f^T & 0 & 0 & 0 & -\mathcal{L}_{bf} \\
0 & \mathbf{C}_f^T & 0 & 0 & 0 & 0 \\
0 & 0 & -\mathcal{L}_{bs}^T & -\mathcal{L}_{bs}^T & 0 & 0
\end{bmatrix}
\begin{Bmatrix} \mathbf{u}_s \\ \mathbf{u}_f \\ \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \boldsymbol{\mu} \\ \mathbf{u}_b \end{Bmatrix}^{n+\frac{1}{2}}
=
\begin{Bmatrix} \mathbf{g}_s \\ \mathbf{g}_f \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}^{n+\frac{1}{2}} \tag{38}$$

$$\begin{aligned}
\mathbf{g}_s^{n+\frac{1}{2}} &= \mathbf{f}_s^{n+\frac{1}{2}} + \mathbf{M}_s(\beta^2\mathbf{u}_s^n + \beta\dot{\mathbf{u}}_s^n), \quad \beta = \frac{1}{\frac{1}{2}\triangle t} \\
\mathbf{g}_f^{n+\frac{1}{2}} &= \mathbf{f}_f^{n+\frac{1}{2}} + \mathbf{M}_f(\beta^2\mathbf{u}_f^n + \beta\dot{\mathbf{u}}_f^n)
\end{aligned}$$

For an efficient solution of the above time-discretized equation, we first solve for $\mathbf{u}_s^{n+\frac{1}{2}}$ and $\mathbf{u}_f^{n+\frac{1}{2}}$ to obtain

$$\begin{aligned}
\mathbf{u}_s^{n+\frac{1}{2}} &= \mathbf{F}_s\,(\mathbf{g}_s^{n+\frac{1}{2}} - \mathbf{B}_s\,\boldsymbol{\lambda}_s^{n+\frac{1}{2}}), \quad \mathbf{F}_s = (\mathbf{K}_s + \beta^2\mathbf{M}_s)^{-1} \\
\mathbf{u}_f^{n+\frac{1}{2}} &= \mathbf{F}_f\,(\mathbf{g}_f^{n+\frac{1}{2}} - \mathbf{B}_f\,\boldsymbol{\lambda}_f^{n+\frac{1}{2}} - \mathbf{C}_f\,\boldsymbol{\mu}_f^{n+\frac{1}{2}}), \quad \mathbf{F}_f = (\mathbf{K}_f + \beta^2\mathbf{M}_f)^{-1}
\end{aligned} \tag{39}$$

Substituting this into the remainder of (38), we obtain the following discrete equation:

$$\begin{bmatrix}
\mathbf{B}_s^T\mathbf{F}_s\mathbf{B}_s & 0 & 0 & \mathcal{L}_{bs} \\
0 & \mathbf{B}_f^T\mathbf{F}_f\mathbf{B}_f & \mathbf{B}_f^T\mathbf{F}_f\mathbf{C}_f & \mathcal{L}_{bf} \\
0 & \mathbf{C}_f^T\mathbf{F}_f\mathbf{B}_f & \mathbf{C}_f^T\mathbf{F}_f\mathbf{C}_f & 0 \\
\mathcal{L}_{bs}^T & \mathcal{L}_{bs}^T & 0 & 0
\end{bmatrix}
\begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \boldsymbol{\mu} \\ \mathbf{u}_b \end{Bmatrix}^{n+\frac{1}{2}}
=
\begin{Bmatrix} \mathbf{B}_s^T\mathbf{F}_s\mathbf{g}_s \\ \mathbf{B}_f^T\mathbf{F}_f\mathbf{g}_f \\ \mathbf{C}_f^T\mathbf{F}_f\mathbf{g}_f \\ 0 \end{Bmatrix}^{n+\frac{1}{2}} \tag{40}$$

*Remark 4:*

- If the shape functions for the fluid displacement satisfies the irrotational constraint a-priori (see, for example, Raviart and Thomas, 1982), the preceding equation becomes

$$\begin{bmatrix} \mathbf{B}_s^T \mathbf{F}_s \mathbf{B}_s & 0 & \mathcal{L}_{bs} \\ 0 & \mathbf{B}_f^T \mathbf{F}_f \mathbf{B}_f & \mathcal{L}_{bf} \\ \mathcal{L}_{bs}^T & \mathcal{L}_{bf}^T & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \mathbf{u}_b \end{Bmatrix}^{n+\frac{1}{2}} = \begin{Bmatrix} \mathbf{B}_s^T \mathbf{F}_s \mathbf{g}_s \\ \mathbf{B}_f^T \mathbf{F}_f \mathbf{g}_f \\ 0 \end{Bmatrix}^{n+\frac{1}{2}} \tag{41}$$

  Thus, the independently assigned localized Lagrange multipliers give rise to uncoupled system dynamic flexibility matrices for the fluid and structure. This leads to a modular implementation for the transient analysis of vibro-acoustic interaction problems. An efficient parallel solution procedure for solving the preceding equation is presented in (Justino, Park and Felippa,1997).

- When the fluid displacement shape functions do not satisfy the irrotational constraint, one can express (40) as

$$\begin{bmatrix} \mathbf{B}_s^T \mathbf{F}_s \mathbf{B}_s & 0 & \mathcal{L}_{bs} \\ 0 & \bar{\mathbf{B}}_f^T \mathbf{F}_f \bar{\mathbf{B}}_f & \bar{\mathcal{L}}_{bf} \\ \mathcal{L}_{bs}^T & \bar{\mathcal{L}}_{bf}^T & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \mathbf{u}_b \end{Bmatrix}^{n+\frac{1}{2}} = \begin{Bmatrix} \mathbf{B}_s^T \mathbf{F}_s \mathbf{g}_s \\ \bar{\mathbf{B}}_f^T \mathbf{F}_f \mathbf{g}_f \\ 0 \end{Bmatrix}^{n+\frac{1}{2}} \tag{42}$$

$$\bar{\mathbf{B}}_f = \begin{bmatrix} \mathbf{B}_f \\ \mathbf{C}_f \end{bmatrix}, \qquad \bar{\mathcal{L}}_{bf} = \begin{bmatrix} \mathcal{L}_{bf} \\ 0 \end{bmatrix}$$

  Clearly, even when the irrotationality constraint is imposed, it does not couple with the structural flexibility matrix, including the fluid-structure boundary structural attributes.

## 6. Vibration Analysis

A vibration analysis can be effected by replacing the time-diferentiation operator ($\frac{d^2}{dt^2}$) in (36) by ($-\omega^2$):

$$\begin{bmatrix} \mathbf{K}_s - \omega^2 \mathbf{M}_s & 0 & \mathbf{B}_s & 0 & 0 & 0 \\ 0 & \mathbf{K}_f - \omega^2 \mathbf{M}_f & 0 & \mathbf{B}_f & \mathbf{C}_f & 0 \\ \mathbf{B}_s^T & 0 & 0 & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \mathbf{B}_f^T & 0 & 0 & 0 & -\mathcal{L}_{bf} \\ 0 & \mathbf{C}_f^T & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{L}_{bs}^T & -\mathcal{L}_{bs}^T & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{u}_f \\ \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \boldsymbol{\mu} \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_s \\ \mathbf{f}_f \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{43}$$

When a modal reduction in the form of

$$\begin{aligned} \mathbf{u}_s &= \mathbf{U}_s \, \mathbf{q}_s \\ \mathbf{u}_f &= \mathbf{U}_f \, \mathbf{q}_f \end{aligned} \tag{44}$$

is introduced, the resulting reduced-basis system becomes

$$\begin{bmatrix} \mathbf{\Lambda}_s - \omega^2 \mathbf{I}_s & 0 & \mathbf{U}_s^T \mathbf{B}_s & 0 & 0 \\ 0 & \mathbf{\Lambda}_f - \omega^2 \mathbf{I}_f & 0 & \mathbf{U}_f^T \mathbf{B}_f & 0 \\ \mathbf{B}_s^T \mathbf{U}_s & 0 & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \bar{\mathbf{B}}_f^T \mathbf{U}_f & 0 & 0 & -\bar{\mathcal{L}}_{bf} \\ 0 & 0 & -\mathcal{L}_{bs}^T & -\bar{\mathcal{L}}_{bf}^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{q}_f \\ \boldsymbol{\lambda}_s \\ \bar{\boldsymbol{\lambda}}_f \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_s^T \mathbf{f}_s \\ \mathbf{U}_f^T \mathbf{f}_f \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{45}$$

$$\mathbf{\Lambda}_s = \mathbf{U}_s^T \mathbf{K}_s \mathbf{U}_s, \quad \mathbf{I}_s = \mathbf{U}_s^T \mathbf{M}_s \mathbf{U}_s$$
$$\mathbf{\Lambda}_f = \mathbf{U}_f^T \mathbf{K}_f \mathbf{U}_f, \quad \mathbf{I}_f = \mathbf{U}_f^T \mathbf{M}_f \mathbf{U}_f$$

Efficient solutions of the preceding partitioned eigenanalysis both on sequential and parallel computers presently constitute a challenging computational mechanics research task.

## 7. Symmetrically Reduced Formulation

As a third specialization of the present general partitioned formulation of vibro-acoustic interaction problem given by (36), we present a symmetrically reduced formulation that involves ($\mathbf{p}$, $\mathbf{u}_s$, $\boldsymbol{\lambda}_s$, $\mathbf{u}_b$), where $\mathbf{p}$ is the pressure in the fluid domain. A motivation for such a formulation is twofold: reduction of the size of the fluid variable and an automatic satisfaction of the irrotationality condition. For various available formulations proposed so far, the reader is referred to Morand and Ohayon (1992).

The starting point for deriving such a reduced formulation is that the discrete pressure can be obtained in terms of the discrete fluid displacement $\mathbf{u}_f$ as

$$p = -\rho_f \, c^2 \, \nabla \mathbf{u}_f \quad \Rightarrow \quad \mathbf{p} = \mathbf{D}_f \, \mathbf{u}_f \tag{46}$$

If this equation is augmented with the irrotational constraints for each fluid element $e$, the resulting equation becomes

$$\begin{Bmatrix} \mathbf{p}^{(e)} \\ \mathbf{0} \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_f^{(e)} \\ \mathbf{C}_f^{(e)} \end{bmatrix} \mathbf{u}_f^{(e)} \tag{47}$$

In general the solution matrix of the above equation does not possess full rank. Note that the elemental fluid displacement $\mathbf{u}_f^{(e)}$ can be obtained as

$$\mathbf{u}_f^{(e)} = \mathbf{D}_p^{(e)} \, \mathbf{p}^{(e)} + \mathbf{R}_f^{(e)} \, \boldsymbol{\alpha}_f^{(e)} \tag{48}$$

where $\mathbf{R}_f^{(e)}$ is the nullspace basis that consists of the elemental rigid-body modes plus any spurious modes present, and $\boldsymbol{\alpha}_f^{(e)}$ is the complementary solution amplitudes. Obviously a desirable discretization is one that engenders no spurious mode. Hence, if there is no spurious mode in the discretization, the size of $\boldsymbol{\alpha}_f^{(e)}$ is at most six. Note also that the size of the pressure vector is substantially smaller than that of the fluid displacement vector.

To obtain the total fluid displacement $\mathbf{u}_f$ from the above equation, we invoke the assembly relation

$$\begin{aligned} \mathbf{u}_f^{(e)} &= \mathbf{L}^{(e)}\ \mathbf{u}_f \\ &\Downarrow \\ \mathbf{u}_f &= \mathcal{L}_f^A\ \mathbf{u}_f^{(e)}, \quad \mathcal{L}_f^A = (\mathbf{L}^{(e)^T}\ \mathbf{L}^{(e)})^{-1}\ \mathbf{L}^{(e)^T} \end{aligned} \tag{49}$$

so that by substituting (48) into the above equation, the fluid displacement $\mathbf{u}_f$ becomes

$$\mathbf{u}_f = \mathbf{D}\ \mathbf{p} + \mathbf{R}_f \boldsymbol{\alpha}_f, \quad \mathbf{D}_p = \mathcal{L}_f^A\ \mathbf{D}_p^{(e)},\ \mathbf{R}_f = \mathcal{L}_f^A\ \mathbf{R}_f^{(e)}\ \bar{1} \quad \text{since} \quad \boldsymbol{\alpha}_f^{(e)} = \boldsymbol{\alpha}_f \tag{50}$$

where $\bar{1}$ stands for column summation, i.e., $\bar{1} = \langle\, 1 \quad 1 \quad \ldots, 1 \,\rangle^T$ .

Therefore, the total degrees of freedom for the fluid domain is reduced to $\frac{1}{3}(n^3+6)$ from $n^3$ for a $(n \times n \times n)$ cube.

When (50) is substituted into (35), the variational functional becomes

$$\begin{aligned} \delta\Pi(\mathbf{u}_s, \mathbf{p}, \boldsymbol{\alpha}_f, \boldsymbol{\lambda}_s, \boldsymbol{\lambda}_f, \mathbf{u}_b, \boldsymbol{\mu}) = {} & \delta\mathbf{u}_s^T\ (\mathbf{K}_s\ \mathbf{u}_s + \mathbf{M}_s\ \ddot{\mathbf{u}}_s - \mathbf{f}_s + \mathbf{B}_s\ \boldsymbol{\lambda}_s) \\ & + \delta\boldsymbol{\mu}^T\ \mathbf{C}_f^T\ (\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f) \\ & + \delta(\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f)^T\ [\mathbf{K}_f\ (\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f) \\ & \qquad + \mathbf{M}_f\ (\mathbf{D}_p\ \ddot{\mathbf{p}} + \mathbf{R}_f\ \ddot{\boldsymbol{\alpha}}_f) - \mathbf{f}_f + \mathbf{C}_f\ \boldsymbol{\mu} + \mathbf{B}_f\ \boldsymbol{\lambda}_f] \\ & + \delta\boldsymbol{\lambda}_\ell^T\ (\begin{bmatrix} \mathbf{B}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f^T \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ (\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f) \end{Bmatrix} - \begin{bmatrix} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{bmatrix} \mathbf{u}_b) \\ & - \delta\mathbf{u}_b^T\ [\ \mathcal{L}_{bs}^T \quad \mathcal{L}_{bf}^T\ ] \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{Bmatrix}, \quad \boldsymbol{\lambda}_\ell = \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{Bmatrix} \end{aligned} \tag{51}$$

By making use of the relations

$$\begin{aligned} \mathbf{C}_f^T \mathbf{D}_p = 0 \quad &\Leftarrow \quad \text{div} \cdot (\text{curl}\ \mathbf{u}_f) = 0 \\ \mathbf{C}_f^T\ \mathbf{R}_f = 0 \quad &\text{since rotation is orthogonal to rigid modes.} \\ \mathbf{K}_f\ \mathbf{R}_f = 0 \quad &\text{as } \mathbf{R}_f \text{ is a nullspace of } \mathbf{K}_f \end{aligned} \tag{52}$$

the foregoing variational functional(51) reduces to

$$\begin{aligned} \delta\Pi(\mathbf{u}_s, \mathbf{p}, \boldsymbol{\alpha}_f, \boldsymbol{\lambda}_s, \boldsymbol{\lambda}_f, \mathbf{u}_b, \boldsymbol{\mu}) = & \\ & \delta\mathbf{u}_s^T\ (\mathbf{K}_s\ \mathbf{u}_s + \mathbf{M}_s\ \ddot{\mathbf{u}}_s - \mathbf{f}_s + \mathbf{B}_s\ \boldsymbol{\lambda}_s) \\ & + \delta\mathbf{p}^T \mathbf{D}_p^T\ [\mathbf{K}_f\ \mathbf{D}_p\ \mathbf{p} + \mathbf{M}_f\ (\mathbf{D}_p\ \ddot{\mathbf{p}} + \mathbf{R}_f\ \ddot{\boldsymbol{\alpha}}_f) - \mathbf{f}_f + \mathbf{B}_f\ \boldsymbol{\lambda}_f] \\ & + \delta\boldsymbol{\alpha}_f^T \mathbf{R}^T\ [\mathbf{M}_f\ (\mathbf{D}_p\ \ddot{\mathbf{p}} + \mathbf{R}\ \ddot{\boldsymbol{\alpha}}_f) - \mathbf{f}_f + \mathbf{B}_f\ \boldsymbol{\lambda}_f] \\ & + \delta\boldsymbol{\lambda}_\ell^T\ (\begin{bmatrix} \mathbf{B}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f^T \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ (\mathbf{D}_p\ \mathbf{p} + \mathbf{R}_f\ \boldsymbol{\alpha}_f) \end{Bmatrix} - \begin{bmatrix} \mathcal{L}_{bs} \\ \mathcal{L}_{bf} \end{bmatrix} \mathbf{u}_b) \\ & - \delta\mathbf{u}_b^T\ [\ \mathcal{L}_{bs}^T \quad \mathcal{L}_{bf}^T\ ] \begin{Bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \end{Bmatrix} \end{aligned} \tag{53}$$

The stationarity of (53) yields the following partition system equation:

$$\begin{bmatrix} \mathbf{K}_s + \frac{d^2}{dt^2}\mathbf{M}_s & 0 & 0 & \mathbf{B}_s & 0 & 0 \\ 0 & \mathbf{K}_p + \frac{d^2}{dt^2}\mathbf{M}_p & \frac{d^2}{dt^2}\mathbf{M}_{pr} & 0 & \mathbf{D}_p^T\mathbf{B}_f & 0 \\ 0 & \frac{d^2}{dt^2}\mathbf{M}_{pr}^T & \frac{d^2}{dt^2}\mathbf{M}_r & 0 & \mathbf{R}_f^T\mathbf{B}_f & 0 \\ \mathbf{B}_s^T & 0 & 0 & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \mathbf{B}_f^T\mathbf{D}_p & \mathbf{B}_f^T\mathbf{R}_f & 0 & 0 & -\mathcal{L}_{bf} \\ 0 & 0 & -\mathcal{L}_{bs}^T & -\mathcal{L}_{bf}^T & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{p} \\ \boldsymbol{\alpha}_f \\ \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_f \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_s \\ \mathbf{D}_p^T\mathbf{f}_f \\ \mathbf{R}^T\mathbf{f}_f \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{54}$$

$$\mathbf{K}_p = \mathbf{D}_p^T\mathbf{K}_f\mathbf{D}_p, \quad \mathbf{M}_p = \mathbf{D}_p^T\mathbf{M}_f\mathbf{D}_p, \quad \mathbf{M}_{pr} = \mathbf{D}_p^T\mathbf{M}_f\mathbf{R}_f, \quad \mathbf{M}_r = \mathbf{R}_f^T\mathbf{M}_f\mathbf{R}_f$$

Of several possible further reductions for vibration analysis, we present the case of eliminating $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}_f$ below. First, from the second row of (54) we obtain

$$\omega^2\boldsymbol{\alpha}_f = -\mathbf{M}_r^{-1}\left[\omega^2\mathbf{M}_{pr}^T\mathbf{p} + \mathbf{R}_f^T(\mathbf{f}_f - \mathbf{B}_f\boldsymbol{\lambda}_f)\right] \tag{55}$$

Substituting this into the fifth equation of (54), we obtain

$$\begin{aligned} &\omega^2\mathbf{B}_f^T\mathbf{D}_p\mathbf{p} - \mathbf{B}_f^T\mathbf{R}\mathbf{M}_r^{-1}\left[\omega^2\mathbf{M}_{pr}^T\mathbf{p} + \mathbf{R}_f^T(\mathbf{f}_f - \mathbf{B}_f\boldsymbol{\lambda}_f)\right] - \omega^2\mathcal{L}_{bf}\mathbf{u}_b = 0 \\ &\qquad\Downarrow \\ &\boldsymbol{\lambda}_f = \mathbf{Q}\left[\mathbf{B}_f^T\mathbf{R}_f\mathbf{M}_r^{-1}\mathbf{R}_f^T\mathbf{f}_f - \omega^2\mathbf{B}_f^T(\mathbf{D}_p - \mathbf{R}_f\mathbf{M}_r^{-1}\mathbf{M}_{pr}^T)\mathbf{p} + \omega^2\mathcal{L}_{bf}\mathbf{u}_b\right] \\ &\mathbf{Q} = \left[\mathbf{B}_f^T\mathbf{R}_f\mathbf{M}_r^{-1}\mathbf{R}_f^T\mathbf{B}_f\right]^{-1} \end{aligned} \tag{56}$$

Equation(54) can now be reduced to read, with $\mathbf{f}_f = 0$ for simplicity, as:

$$\begin{bmatrix} \mathbf{K}_s - \omega^2\mathbf{M}_s & 0 & \mathbf{B}_s & 0 \\ 0 & \mathbf{K}_p - \omega^2\hat{\mathbf{M}}_p & 0 & \omega^2\hat{\mathcal{L}}_{bf} \\ \mathbf{B}_s^T & 0 & 0 & -\mathcal{L}_{bs} \\ 0 & \omega^2\hat{\mathcal{L}}_{bf}^T & -\mathcal{L}_{bs}^T & -\omega^2\mathcal{L}_{bf}^T\mathbf{Q}\mathcal{L}_{bf}^T \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{p} \\ \boldsymbol{\lambda}_s \\ \mathbf{u}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_s \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} &\hat{\mathbf{D}}_p = \mathbf{D}_p - \mathbf{R}_f\mathbf{M}_r^{-1}\mathbf{M}_{pr}^T \\ &\hat{\mathbf{M}}_p = \mathbf{M}_p - \mathbf{M}_{pr}\mathbf{M}_r^{-1}\mathbf{M}_{pr}^T - \hat{\mathbf{D}}_p^T\mathbf{B}_f\mathbf{Q}\mathbf{B}_f^T\hat{\mathbf{D}}_p \\ &\hat{\mathcal{L}}_{bf} == \hat{\mathbf{D}}_p^T\mathbf{B}_f\mathbf{Q}\mathcal{L}_{bf} \end{aligned} \tag{57}$$

It should be noted that a further reduction is possible, for example, the elimination of the interface boundary displacement $\mathbf{u}_b$. However, this leads to an unsymmetric formulation.

## 8. Discussions

The present paper has presented a localized Lagrange multipliers method for the modeling of vibro-acoustic interaction problems with the displacement variables both for the structure and fluid. A

two-spring model is used to illustrate the basic features of the localized Lagrange multipliers method and compared with the classical Lagrange multipliers method.

A variational formulation of vibro-acoustic problems is introduced to model small-amplitude fluid motions and a linear structural system in terms of their displacements. The total system is then discretized as one entity. Partition of the fluid domain and the structural domain is realized employing the localized Lagrange multipliers method. It is shown that the partitioned structural and fluid equations are coupled through the partition boundary global displacement vector that are common both to the fluid and structure, and through the sum of the localized Lagrange multipliers via Newton's third law.

The partitioned vibro-acoustic equations are then specialized to three potential applications: implicit-implicit transient analysis, vibration analysis using substructuring methods, and a symmetric reduction to structural displacement and fluid pressure formulation.

Additional formulations can be derived from the present variational formulation, which are presently under investigation and will be reported in the future.

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