Brief Paper

Designing robust sliding hyperplanes for parametric uncertain systems: a Riccati approach

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Abstract

In this paper, we propose a method to design robust sliding hyperplanes in the presence of mismatched parametric uncertainty based on quadratic stability. The robust sliding hyperplane is constructed from a Riccati inequality associated with quadratic stabilizability. The proposed method enables us to deal with structured uncertainty and optimize the sliding motion by applying the guaranteed cost control idea.

Keywords: Sliding mode control; Quadratic stability; Guaranteed cost control; Mismatched parametric uncertainty

1. Introduction

An advantage of sliding mode control is its robustness to matched disturbances in the sliding mode. These matched disturbances may represent parametric uncertainty or external disturbances which are in the range space of the input matrix. In the literature concerning sliding mode control the matching condition on disturbances or uncertainties is a main assumption. For such systems, sliding modes have been developed by many researchers (Utkin & Yang, 1979; Dorling & Zinober, 1986; Young & Özgüner, 1993). However, the matching condition is restrictive and inadequate to modeling uncertainty in some mechanical systems such as the flexible beam pointing systems and the flexible robot arm systems as can be seen in Fig. 1(a). A torque actuator is located at the joint where friction force exists and the beam with the negligible internal damping is so flexible that we should consider the modal behavior at least up to the third mode including the rigid mode for stabilizing the line of direction. Fig. 1(b) shows the mass–spring system which models the system in Fig. 1(a). Note that the friction disturbance satisfies the matching condition but not the uncertainties in the modal parameters such as natural frequencies. Recently, to treat such uncertain systems, efforts have been made both in sliding mode control and min–max approach. For example, sliding mode control with parameter adaptation (Kwan, 1995; Tunay & Kaynak, 1995; Phadke, 1996 and references therein) or eigenvalue assignment method (Dorling & Zinober, 1988), and min–max approaches (Najson & Kreindler, 1996) are those. Note that min–max approaches are different from the usual sliding mode control specially in the convergence behavior around the switching function. We refer to DeCarlo, Zak & Matthews (1988) for detailed comparison.

In this paper, we propose a different way of sliding mode design motivated by quadratic Lyapunov function approaches. It has been shown (Su, Drakunov & Özgüner, 1996) that a Lyapunov equation (or the algebraic Riccati equation in LQR problems) can be adopted to yield appropriate sliding modes for matched uncertainty. However, there are few researches on mismatched uncertainty. The purposes of this paper are to (i) exploit the connection between quadratic stability and the sliding mode, and (ii) assess robust performance as well as robust stability in the sliding mode. To this end, the basic
idea is to combine the recently established quadratic stabilization methods (Petersen, 1987; Khargonekar, Petersen & Zhou, 1990; Zhou, Khargonekar, Stoustrup & Niemann, 1995) with the design of robust sliding hyperplanes. It will be shown that robust sliding hyperplanes are constructed using a Riccati inequality, which suffices quadratic stabilizability via linear full state feedback for uncertain systems. Thanks to the property of the quadratic stabilization methods, several issues such as back for uncertain systems. Thanks to the property of the idea is to combine the recently established quadratic stabilization methods (Petersen, 1987; Khargonekar, Petersen & Zhou, 1990; Zhou, Khargonekar, Stoustrup & Niemann, 1995) with the design of robust sliding hyperplanes. It will be shown that robust sliding hyperplanes are constructed using a Riccati inequality, which suffices quadratic stabilizability via linear full state feedback for uncertain systems. Thanks to the property of the quadratic stabilization methods, several issues such as back for uncertain systems. Thanks to the property of the

Most of notations are fairly standard. Among them, $\| \cdot \|$ denotes the Euclidean norm for a vector (or the matrix induced norm for a matrix), and inequality signs for matrices are used to express sign-definiteness of symmetric matrices. Also, $\sigma_{\max}(\cdot)$ (or $\sigma_{\min}(\cdot)$) denotes the maximum (or minimum) singular value of the argument matrix.

2. Notations and problem description

Consider the following uncertain linear systems described by

$$\dot{x} = (A + \Delta A)x + Bu + Fw,$$

(1)

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $w \in \mathbb{R}^d$ is the external disturbance. It is assumed that the pair $(A, B)$ is controllable and $\text{rank}(B) = m$. Note that disturbances satisfy the matching condition while uncertainty does not. Let us assume that the magnitude bound of each component in $w$ is known as $w_i(t) \leq \tilde{w}_i(t)$ for $i = 1, \ldots, l$. The uncertainty has the form

$$\Delta A = \sum_{i=1}^{p} \delta_i(t)E_i, \quad |\delta_i(t)| \leq 1$$

(2)

for $i = 1, \ldots, p$, where constant matrices $E_i$’s are known and $\text{rank}(E_i) = q_i$, and $\delta_i(\cdot)$’s are Lebesgue-measurable. In general, the assumed form of uncertainty can be decomposed as follows:

$$\Delta A = MDN,$$

(3)

where $D = \text{blockdiag}\{\delta_1(t)I_{q_1}, \ldots, \delta_p(t)I_{q_p}\} \in \mathbb{R}^{h \times h}$ and, $M \in \mathbb{R}^{n \times h}$ and $N \in \mathbb{R}^{h \times n}$ are constant. For future reference, we define the set of scales as

$$S_D = \{ Y | Y = \text{blockdiag}[Y_1, \ldots, Y_p], \quad 0 < Y_i = Y_i^T \in \mathbb{R}^{h \times s_i} \}. $$

(4)

Note that $X^{1/2}DX^{-1/2} = D$ for any $X \in S_D$. In practice, transforming (2) into (3) is not unique, however, it does not matter because the scales will be left as design parameters. Such characterizations have been used for handling real parametric uncertainties (Packard, Zhou, Pandey, Leonhardson & Balas, 1992; Zhou, Doyle & Glover, 1996).

To have a regular form of system (1), a nonsingular matrix $T$ can be always chosen such that

$$TB = \begin{bmatrix} 0_{(a-m) \times n} \\ B_2 \end{bmatrix},$$

where $B_2 \in \mathbb{R}^{m \times m}$ is nonsingular. For the ease of explanation, let us choose

$$T = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix},$$

where $U_1 \in \mathbb{R}^{(a-m) \times n}$ and $U_2 \in \mathbb{R}^{m \times n}$ are the sub-blocks of a unitary matrix obtained from the singular value decomposition of $B$, i.e.,

$$B = [U_1, U_2] \Sigma [0_{(a-m) \times m} \Sigma]^{-T},$$

By the state transformation using $\xi = Tx$, the systems in the regular form are

$$\dot{\xi} = (\bar{A} + \bar{\Delta}A)\xi + \bar{B}(u + Fw),$$

(5)
where $\bar{A} = TAT^{-1}$, $\Delta \bar{A} = T\Delta AT^{-1}$ and $\bar{B} = TB$. Or, equivalently,
\[
\begin{align*}
\dot{\xi}_1 &= (\bar{A}_{11} + \Delta \bar{A}_{11})\xi_1 + (\bar{A}_{12} + \Delta \bar{A}_{12})\xi_2, \\
\dot{\xi}_2 &= (\bar{A}_{21} + \Delta \bar{A}_{21})\xi_1 + (\bar{A}_{22} + \Delta \bar{A}_{22})\xi_2 \\
&\quad + B_2(u + Fw),
\end{align*}
\]
where $\xi_1 \in \mathbb{R}^{m-n}$, $\xi_2 \in \mathbb{R}^m$, and $B_2 = \Sigma V^T$. Note $\Delta \bar{A}_{11} = U^T_1 M D N U_1$ and $\Delta \bar{A}_{12} = U^T_2 M D N U_1$. It is interesting that $\Delta \bar{A}_{11}$ and $\Delta \bar{A}_{12}$ are dependent to each other. Without loss of generality, suppose that a sliding hyperplane is
\[
s(t) = C\xi_1 + \xi_2 = 0,
\]
where $C \in \mathbb{R}^{m \times (n-m)}$. Hence, substituting $\xi_2 = -C\xi_1$ to (6) gives the sliding motion
\[
\dot{\xi}_1 = (\bar{A}_{11} + \Delta \bar{A}_{11} - (\bar{A}_{12} + \Delta \bar{A}_{12})C)\xi_1.
\]
If the uncertainty were matched, $\Delta \bar{A}_{11}$ and $\Delta \bar{A}_{12}$ may not appear in (8) so that the stabilizing $C$ can be always found under the controllability of $(\bar{A}_{11}, \bar{A}_{12})$ (Utkin & Yang, 1979). From the above, the design of a robust sliding mode control is possible if (i) there exists a $C \in \mathbb{R}^{m \times (n-m)}$ which guarantees robust stability of (8), and (ii) there exists a control law which makes the sliding function asymptotically stable for a specified sliding hyperplane. The remainder of this note is devoted to the design of a sliding hyperplane and control law to satisfy these requirements.

3. Main results

We start with the known results in quadratic stability for uncertain linear systems. For the readability of this manuscript, we summarize the works by Petersen (1987), Khargonekar, Petersen and Zhou (1990), and Zhou et al. (1995) in the following.

**Definition 1.** System (1) with a feedback controller $u = q(t, x)$ is said to be quadratically stable if there exists a $P > 0$ such that $V(t, x) = x^TPx$ is a Lyapunov function for the closed-loop system.

**Lemma 1.** Suppose $w = 0$. Then, system (1) is quadratically stabilizable via linear full state feedback if there exist some constant matrices $P > 0$, $X \in S_D$ and $K \in \mathbb{R}^{m \times n}$ satisfying
\[
(A - BK)^TP + P(A - BK) + PMXM^TP + N^TX^{-1}N < 0.
\]

**Proof.** Let the control be given by $u = -Kx$. Then, it is straightforward to show quadratic stability of the closed-loop systems by applying the bounding technique $HY + Y^TH \leq HH^T + Y^TY$ to uncertain terms. That is,
\[
P\Delta A + \Delta A^TP = P(MDN) + (MDN)^TP
\]
\[
= (PMX^{1/2})(DX^{-1/2}N)
\]
\[
+ (DX^{-1/2}N)(PMX^{1/2})^T
\]
\[
\leq PMXM^TP + N^TX^{-1}N
\]
for any $X \in S_D$. □

**Remark 1.** The scales play an important role to express all possible bounding functions due to non-unique choice of $M$ and $N$ in (10). Consequently, the inclusion of scales enlarges the feasibility of the Riccati inequality (9) and reduces design conservatism.

**Remark 2.** The feasibility of the Riccati inequality (9) is equivalent to the well known scaled small gain condition for multi-block uncertainty (Zhou et al., 1996). The inequality can be converted into a linear matrix inequality (LMI) by using Schur complement and the change of variable such that $K := KP^{-1}$. Hence, the feasible solutions can be globally found by the LMIs method (Boyd, Ghaoui, Feron & Balakrishnan 1994; Gahinet, Nemirovski, Laub & Chilali, 1995).

Now, given a $Q \geq 0$, let us define the set of positive-definite matrices as
\[
\Xi(Q) := \left\{ \begin{array}{l}
(A - BK)^TP + P(A - BK) + PMXM^TP \\
\quad + N^TX^{-1}N + Q < 0,
\end{array} \right\}
\]
\[
P > 0
\]
\[
\text{for some } K \in \mathbb{R}^{m \times n} \text{ and } X \in S_D.
\]

Since the existence of positive-definite matrix $P$ does not depend on the choice of $Q \geq 0$, determining if $\Xi(Q)$ is non-empty is equivalent to the quadratic stabilizability problem in Lemma 1.

The following theorem is the main result of this paper.

**Theorem 1.** Given a $Q \geq 0$, suppose that $\Xi(Q)$ is non-empty. Then, the reduced system (8) is quadratically stable with $C = (U^T_1 P U_1)^{-1}U^T_2 P U_2$ for any $P \in \Xi(Q)$. In this case, the sliding function is $s(t) = Gx$ and asymptotically converges to zero by the control
\[
u = \begin{cases}
0, & \|s(t)\| = 0, \\
-(\Xi V^T)^{-1}(GAx + \beta s + Z(t)\text{sign}(s)), & \|s(t)\| > 0,
\end{cases}
\]
\[
\beta > 0, \quad \text{sign}(s) = [\text{sign}(s_1), \ldots, \text{sign}(s_m)]^T, \quad G = (U^T_1 P U_1)^{-1}U^T_2 P \quad \text{and} \quad Z(t) = \text{diag}([z_1(t), \ldots, z_m(t)], [z_2(t), \ldots, z_m(t)], \ldots, [z_1(t), \ldots, z_m(t)].
\]
where
\[ z_i(t) = \sum_{j=1}^{k} ||(GM)_{ij}(Nx_j)|| + \sum_{k=1}^{m} ||(GBF)_{ik}\tilde{w}_k(t)||. \]  \hspace{1cm} (13)

**Proof.** For convenience, system (8) is equivalently redefined as follows:
\[ \dot{\xi}_1 = \{\dot{A}_{11} - \dot{A}_{12} C + M_rDN_x\}\xi_1, \]  \hspace{1cm} (14)
where \( M_r = U_1^T M, \quad N_x = NU_2 - NU_1 C. \) Consider a \( P \in \Xi(Q). \) Then, there exist \( K \in \mathbb{R}^{m \times n} \) and \( W \in S_D \) such that
\[ (A - BK)^TP + P(A - BK) + PMWMT^TP + N^TW^{-1}N < 0. \]  \hspace{1cm} (15)

Using
\[ T = \begin{bmatrix} U_2^T \\ U_1^T \end{bmatrix} \text{ and } T^{-1} = [U_2, U_1], \]
let us denote some matrices as follows:
\[ \bar{A}_c = T(A - BK)T^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} - B_2 K U_2 & \bar{A}_{22} - B_2 K U_1 \end{bmatrix}, \]
\[ \bar{P} = T^{-1}PT^{-1} = \begin{bmatrix} U_1^TPU_2 & U_1^TPU_1 \\ U_2^TPU_2 & U_2^TPU_1 \end{bmatrix} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}. \]  \hspace{1cm} (16)

Then, (15) can be rewritten as
\[ \bar{A}_c^TP + P\bar{A}_c + PMWMT^TP + T^{-1}N^TW^{-1}NT^{-1} < 0. \]  \hspace{1cm} (17)

By pre- and post-multiplying \([I_{n-m}, - P_{12} P_{22}^{-1}]\) and \([I_{n-m}, - P_{12} P_{22}^{-1}]^T, \) respectively, (17) can be represented as
\[ (\bar{A}_{11} - \bar{A}_{12} P_{22}^{-1} P_{12})^TP_r + P_r(\bar{A}_{11} - \bar{A}_{12} P_{22}^{-1} P_{12}) + P_r M_r W M_r^T P_r + (NU_2 - NU_1 P_{22}^{-1} P_{12})^TW^{-1} \times (NU_2 - NU_1 P_{22}^{-1} P_{12}) < 0, \]  \hspace{1cm} (18)
where \( P_r = \bar{P}_{11} - \bar{P}_{12} P_{22}^{-1} \bar{P}_{12}. \) Note that \( P_r > 0 \) because \( P > 0. \) Therefore, by choosing \( C = P_{22}^{-1} P_{12}, \) (18) yields
\[ (\bar{A}_{11} - \bar{A}_{12} C)^TP_r + P_r(\bar{A}_{11} - \bar{A}_{12} C) + P_r M_r W M_r^T P_r + N_r^TW^{-1}N_r < 0, \]
which implies quadratic stability of the reduced system (14) by Lemma 1.

For such a choice of \( C, \) the sliding function is
\[ s(t) = [(P_{22}^{-1} P_{12}^T, 1_{m \times n})] Tx = (P_{22}^{-1} U_1^TI_1 U_2^T + U_1^T)x = P_{22}^{-1} U_1^T U_2^T + U_1^T U_1^T x = P_{22}^{-1} U_1^T P x. \]  \hspace{1cm} (19)

To prove the asymptotic stability of the sliding function, define a Lyapunov function candidate as \( V_s = 0.5s^2. \) By noting \( GB = (U_1^T U_1)^{-1} U_1^T P U_1 \Sigma V^T = \Sigma V^T, \) it follows that
\[ \dot{V}_s = GAx + GMDNx + GBFW + \Sigma V^T u. \]  \hspace{1cm} (20)

Hence, when \( ||s|| \neq 0, \)
\[ \dot{V}_s = -\beta s^T s + \sum_{i=1}^{m} \left( \sum_{j=1}^{n} s_i(GM)_{ij} \delta_j(Nx_j) \right) \]
\[ + \sum_{k=1}^{m} \left( \sum_{i=1}^{n} ||(GM)_{ik}(Nx_i)|| \right) \]
\[ \leq -\beta s^T s + \sum_{i=1}^{m} \left( \sum_{j=1}^{n} ||(GM)_{ij}(Nx_j)|| \right) \]
\[ + \sum_{k=1}^{m} ||(GBF)_{ik}|| \tilde{w}_k(t) - z_i(t)||s|| \]
\[ = -\beta ||s||^2. \]  \hspace{1cm} (21)

Therefore, the sliding function \( s(\cdot) \) asymptotically converges to zero. This completes the proof. \( \Box \)

It is noted that the minimum convergence speed of the sliding function is determined by \( \beta \) as can be seen in (21). The larger the value of \( \beta, \) the faster the convergence of the norm of sliding function.

In the absence of uncertainty, i.e., in the case of \( M = N = 0, \) our result is consistent with that by Su et al. (1996) because the Riccati inequality in (11) reduces to a Lyapunov inequality and the sliding function can be rewritten as
\[ s(t) = (U_1^T P U_1)^{-1} U_1^T P x \]
\[ = (U_1^T P U_1)^{-1} \Sigma^{-1} V^T B^T P x = L B^T P x, \]  \hspace{1cm} (22)
where \( L = (U_1^T P U_1)^{-1} \Sigma^{-1} V^T, \) which is invertible. Hence, our results can be viewed as an extension of Su, Drakunov and Özgüner (1996) to the parametric uncertain systems.
Regarding the unstructured uncertainty, Theorem 1 is parallel to the min−max control approach by Najson and Kreindler (1996) in that a switching function is designed by utilizing a Riccati equation (or inequality) that guarantees quadratic stability. However, the two methods are completely different in many aspects such as the control derivation procedure and the regulation behaviors of closed-loop systems. Moreover, the Riccati equation (equation (75) of Najson and Kreindler (1996)) is a special case of the Riccati inequality (11). In case of unstructured uncertainty, without loss of generality, we can suppose that \( Q = I, M = N = \sqrt{\mu}I \) (i.e., \( \sigma_{\max}(\Delta A) \leq \mu \)) and \( X = \eta I \) for some \( \eta > 0 \) in (11). Then, if we assumed the specific structure of \( K \) such that \( K = 0.5B^TP \), feasibility of the Riccati inequality (11) would become equivalent to that of Eq. (75) of the reference.

Now the closed-loop characteristics during the so-called reaching phase (i.e., while the sliding function converges to zero) is of concern. Suppose that control (12) is applied to system (5). Using the relation \( \xi_2 = s(t) - C\xi_1 \) in (7), the reduced dynamics can be rewritten as

\[
\dot{\xi}_1 = (\dot{A}_{11} + \Delta \dot{A}_{11} - (A_{12} + \Delta A_{12})C)\xi_1 + (\dot{A}_{12} + \Delta \dot{A}_{12})s(t).
\]

Note that \( C \) quadratically stabilizes the above system when \( s(t) = 0 \). Through elaborate manipulations using the property of quadratic stability, it can be shown that

\[
||\xi_1(t)||^2 \leq z||s(t)||^2 + k\xi_1(0), 0(0)
\]

for some constants \( z \) and \( k(\cdot, \cdot) \) depending on the initial values (see the appendix for detailed descriptions). Also, we have

\[
||\xi_2(t)|| = ||s(t) - C\xi_1(t)|| \leq ||s(t)|| + \sigma_{\max}(C)\sqrt{z}||s(t)||^2 + k(\xi_1(0), 0(0)).
\]

Therefore, from (24) and (25), the Euclidean norm of the closed-loop state vector should be bounded while \( ||s(t)|| \) approaches to zero.

Besides the stability issue in the above, Theorem 1 makes it possible to assess the robust performance issue. To this end, consider a \( P \in \Xi(Q) \), which is the solution matrix with a certain \( (K, X) \in R^{n \times n} \times S_{+} \). By rewriting (1) as

\[
\dot{x} = (A - BK + \Delta A)x + B\psi,
\]

where \( \psi = u + Kx + Fw \), the derivative with respect to time, on the sliding hyperplane, for the quadratic function defined as \( V_x = \dot{x}^TPx \), can be shown as

\[
\dot{V}_x < -\dot{x}^TQx + 2\dot{x}^TPB\psi = -\dot{x}^TQx
\]

because \( PBx = 0 \) on \( s = LB^TPx = 0 \). By integrating both sides of (27), it follows that

\[
\int_{t_i}^{t_f} \dot{x}^TQx \ dt < \sigma_{\max}(P)||x(t_i)||^2,
\]

where \( t_i \) is the initial time at which the sliding motion occurs. Therefore, after choosing a non-negative definite \( Q \) as a state weighting matrix, \( X \) and \( K \) are optimized to minimize the maximum singular value of \( P \) satisfying the Riccati inequality in (11). Such a concept is called guaranteed cost control (Chang & Peng, 1972).

In summary, combining quadratic stabilization method and sliding mode control is advantageous in various aspects. First, the characteristics of quadratic stabilization such as robustness to mismatched uncertainty, robust performance and scales for the structured uncertainty can be easily taken into account for sliding mode design. Second, design procedures are very simple since solving quadratic stabilizability is enough to have a robust sliding mode controller. Moreover, we do not need any complicated methods such as parameter adaption algorithms to obtain robustness to mismatched uncertainty. However, regardless of the aforementioned advantages, the application to systems with large variation of uncertainty remains as a further study. Note that the Riccati inequality may not give a feasible solution for a class of large uncertainties. In such cases, adaptive techniques may be more effective than the proposed method.

4. An illustrative example

We consider the mass−spring system shown in Fig. 1(b). The nominal stiffness of springs is 3.3 N/m and the masses are unit. The model uncertainty of stiffness is assumed to be within 50% from the nominal value. Therefore, when the state vector is defined as \( x = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3]^T \), the uncertainties in the form of (2) are given as

\[
E_1 = \begin{bmatrix} 0_{3 \times 6} \\ -1.65 & 1.65 & 0 & 0 & 0 & 0 \\ 1.65 & -1.65 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
E_2 = \begin{bmatrix} 0_{4 \times 6} \\ 0 & -1.65 & 1.65 & 0 & 0 & 0 \\ 0 & 1.65 & -1.65 & 0 & 0 & 0 \end{bmatrix}
\]

By using the minimal rank decomposition method, the above uncertainty can be decomposed as

\[
M = \begin{bmatrix} 0_{3 \times 2} \\ 0 & \sqrt{1.65} & 0 \\ \sqrt{1.65} & -\sqrt{1.65} & 0 \end{bmatrix}
\]
The obtained results are as follows:

\[
N = \begin{bmatrix}
\sqrt{1.65} & \sqrt{1.65} & 0 \\
0 & \sqrt{1.65} & -\sqrt{1.65} \\
\delta_1(t) & 0 & \delta_2(t)
\end{bmatrix}_{2 \times 3},
\]

Then, we used the LMI ToolBox of MATLAB (Gahinet et al., 1995) to solve an optimization problem: minimize \(\sigma_{\text{max}}(P)\), with respect to \(K\) and \(X\), subjected to the Riccati inequality (11) for \(Q = \text{diag}[0,0.5,1,0,0.3,0.5] \times 10^{-6}\). Note that (11) should be formulated into LMIs by using the change of variables and the Schur complement to use the LMI solver. The obtained results are as follows:

\[
\sigma_{\text{max}}(P) \leq 2.768 \times 10^{-5},
\]

\[
P = \begin{bmatrix}
0.4274 & 1.346 & -1.524 & 0.0205 & 0.7879 & 0.3980 \\
1.346 & 15.11 & -10.91 & 0.0595 & 4.313 & 5.619 \\
-1.524 & -10.91 & 11.54 & -0.0720 & -4.157 & -2.671 \\
0.0205 & 0.0595 & -0.0720 & 0.0024 & 0.0413 & 0.0160 \\
0.7879 & 4.313 & -4.157 & 0.0413 & 2.165 & 1.291 \\
0.3980 & 5.619 & -2.671 & 0.0160 & 1.291 & 4.011
\end{bmatrix} \times 10^{-6},
\]

\[X = \begin{bmatrix}
2.311 \times 10^6 & 0 \\
0 & 2.497 \times 10^1
\end{bmatrix},
\]

\[K = \begin{bmatrix}
197.6, 488.1, -623.1, 9.814, 320.4, 137.9
\end{bmatrix}.
\]

Hence, we have

\[C = \begin{bmatrix}
24.86, & -30.12, & -8.554, 17.25, 6.681
\end{bmatrix},
\]

\[G = \begin{bmatrix}
8.554, 24.86, & -30.12, 1.000, 17.25, 6.681
\end{bmatrix}.
\]

Fig. 2 shows the simulation results for the nominal system and the perturbed systems in case of \(\beta = 0.5\). This example shows that the conventional scheme of sliding mode control can be successfully applied without any modification such as combining with adaptive techniques to some uncertain systems as long as the sliding hyperplane and the control law can be robustly designed.

5. Concluding remarks

In this note, we proposed a method to design robust sliding hyperplanes based on quadratic stability. Quadratic stabilizability described by a Riccati inequality suffices the existence of robust sliding hyperplanes. The proposed design procedure is very simple and systematic in the optimization framework. Most importantly, it was shown that the sliding motion can be optimized in the sense of guaranteed cost control approach based on quadratic stability.

**Appendix A. Details for showing the norm-boundedness of \(\xi_1\)**

For the simplicity of notations, let (23) be denoted as

\[
\dot{\xi}_1 = (\bar{\Delta}_{cl,r} + \Delta \bar{A}_{cl,r})\xi_1 + (\bar{A}_{12} + \Delta \bar{A}_{12})\xi(t),
\]

where \(\bar{\Delta}_{cl,r} = \bar{A}_{11} - \bar{A}_{12}C\) and \(\Delta \bar{A}_{cl,r} = \Delta \bar{A}_{11} - \Delta \bar{A}_{12}C\). Since \(C\) has been chosen to quadratically stabilize the sliding motion (8), there exist some \(\varepsilon > 0\) and \(P_r > 0\) satisfying

\[
\bar{A}_{cl,r}^T P_r + P_r \bar{A}_{cl,r} + \bar{\Delta} \bar{A}_{cl,r}^T P_r + P_r \Delta \bar{A}_{cl,r} + \varepsilon I \leq 0.
\]

(A.1)

Then, for a quadratic function \(V_\xi = \xi_1^T P_r \xi_1\), the time derivative is as follows:

\[
\dot{V}_\xi = \xi_1^T (\bar{A}_{cl,r}^T P_r + P_r \bar{A}_{cl,r} + \Delta \bar{A}_{cl,r}^T P_r + P_r \Delta \bar{A}_{cl,r}) \xi_1
\]

\[
+ 2 \xi_1^T P_r (\bar{A}_{12} + \Delta \bar{A}_{12}) s(t)
\]

\[
\leq - \varepsilon \|\xi_1\|^2 + \gamma \xi_1^T \xi_1
\]

\[
+ \frac{1}{\gamma} (s(t))^T (\bar{A}_{12} + \Delta \bar{A}_{12})^T P_r (\bar{A}_{12} + \Delta \bar{A}_{12}) s(t)
\]

\[
\leq - (\varepsilon - \gamma) \|\xi_1\|^2 + \frac{\mu}{\gamma} \|s(t)\|^2
\]

for any \(\gamma > 0\), where \(||(\bar{A}_{12} + \Delta \bar{A}_{12})^T P_r (\bar{A}_{12} + \Delta \bar{A}_{12})|| \leq \mu\). Choose a \(\gamma > 0\) such that \(\gamma = \varepsilon\), integrate both sides to have

\[
\|\xi_1(t)\|^2 \leq \frac{1}{\sigma_{\text{min}}(P_r)} \left( \xi_1(0)^T P_r \xi_1(0) + \int_0^t \|s(r)\|^2 \, dr \right)
\]

(A.2)

Also, using (21), we have

\[
\int_0^t \|s(r)\|^2 \, dr \leq \frac{1}{2\beta} (\|s(0)\|^2 - \|s(t)\|^2).
\]

(A.3)
Hence, it follows, from (A.2) and (A.3), that

\[ \| \xi_1(t) \|^2 \leq -\frac{\mu}{2\epsilon\sigma_{\min}(P_r)}\| s(t) \|^2 + \frac{1}{\sigma_{\min}(P_r)}(\xi_1(0))^T P_r \xi_1(0) \]

\[ + \frac{\mu}{2\epsilon\beta}\| s(0) \|^2, \]

which shows (24).

References


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