New approximations of external acoustic–structural interactions: Derivation and evaluation

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**A R T I C L E   I N F O**

Article history:
Received 8 June 2008
Received in revised form 4 December 2008
Accepted 5 December 2008
Available online 24 December 2008

Keywords:
External acoustics
Structure–acoustic interactions
Retarded and advanced acoustic potential

**A B S T R A C T**

New approximate models for external acoustics interacting with flexible structures are developed. The basic form of the present models is obtained by a combination of the Laplace-transformed retarded and advanced potentials with the weighting parameter as part of the model equation. It is shown that the maximum attainable time-derivative of convergent approximate models is two, hence any attempt to include higher orders will lead to non-convergent models. The present external acoustic model is implemented and interfaced with a finite element structural analyzer for the transient response analysis of submerged spherical and cylindrical shells subjected to a series of incident waves. Comparisons of the present results with the classical analytical solutions and the Doubly Asymptotic Approximations (DAAs) show that proposed model offers improved accuracy especially for early-time responses, exhibits computational robustness, and maintains the impulse response consistency that are desirable for inverse problems. The present model is applied for a shock response correlation of an experimental complex ring cylinder, which demonstrates the applicability of the proposed approximate model to practical engineering problems.

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1. Introduction

Computationally tractable models of external acoustic fields interacting with flexible structures have received intense interest over the past three decades. A dominant approach adopted in the development of external acoustic–structure interaction models has been to improve the classical plane wave approximation models\([1–6]\). A computational advantage of these models accrues from their implementation involving boundary integrals evaluated only over the interaction surfaces. It should be mentioned that, in a parallel effort, the finite element-based approach has also been pursued. As our focus is on the use of Kirchhoff’s pressure retarded potential for constructing approximate interaction models, we will not dwell on the finite element models that require discretization of the pressure field volumes and not interaction surfaces, and instead we refer to a recent excellent article\([7]\) and references therein. Regardless of interaction models one may adopt, the fidelity of various approximate interaction models are evaluated by comparing the results obtained by the approximate models with the corresponding analytical solutions that are obtained either by the retarded acoustic potential equation or the continuum wave equation\([3,8]\).

One of the computationally tractable boundary element-based approximate acoustic–structure interaction equations is the Doubly Asymptotic Approximations (DAAs) proposed by Geers and co-researchers\([4]\). In deriving their DAA models, the two limiting cases have been modeled: early-time approximation (ETA) and late-time approximation (LTA) by employing the initial-value and final-value theorems of the Laplace transform to the series expansion terms of the Kirchhoff’s spherical acoustic integral wave equation\([5]\). The first or second-order DAA models are then constructed by matching the model impedances with the ETA and LTA limits. The DAA models have proved to be adequate for characterizing the fluid acoustic radiation damping affecting the structural responses that are dominated by low-frequency components. Consequently, they have been used as a production external acoustic analyzer that is interfaced with several commercial structural analysis codes\([9]\).

When one focuses on the pressure field modeling, not only the pressure magnitudes but also its phase information have to be obtained accurately. This is especially true for identifying sound sources as well as sound intensities. For example, a careful examination of most existing approximate models for ideal structural geometries, when compared with the exact models obtained via Kirchhoff’s formula, reveals that the dominant acoustic scattering pressure modes are often not represented. Hence, most existing approximate models, while adequate for structural response...
calculations, may not be applicable to inverse acoustics problems wherein the primary objective is to identify the sound sources.

This has motivated the present authors to develop acoustic models that can capture predominant acoustic modes, as distinct from structural modes, and yet that are computationally attractive. From the theoretical point of view, the well-known Kirchhoff’s retarded potential equation may be considered as the foundation of all the existing approximate models. Under this premise, different approximate acoustic models originate from the different approximations of the retarded (or delayed) operator. As shown in the present paper, a straightforward derivation of a second-order approximate pressure equations from the retarded potential leads to unstable models, unless one invokes an elaborate procedure akin to the derivation of the second-order DAA2. A careful examination of the DAA2, however, revealed that the discrete DAA2 model fails to represent the impulse impedance. The spherical modal version of the DAA2 does possess the correct initial impulse impedance; however, it is not applicable to general interaction surfaces as it is a spherical modal model. We hold the view that the correct impulse impedance property is an important desirable property if approximate pressure interaction models are to be adopted for acoustic pressure identification applications. As detailed in the next section, after an exhausting series of attempts the present authors have concluded that it would be unlikely, if not impossible, to develop stable, convergent and high-order accurate pressure approximations employing the retarded pressure potential alone. This reinforces the inventiveness of the doubly asymptotic matching procedure that had been so successfully exploited by Geers and his colleagues [4].

To break this impasse, viz., the inability to develop stable, more accurate and computationally tractable pressure approximations from the retarded potential alone, we have made a key departure from conventional paradigm. In the present paper, we employ a weighted combination of the retarded potential and the advanced potential, and a precursor to the present improved models was presented in [10,11]. In employing the advanced potential, we are keenly mindful of the disagreement between Ritz and Einstein [12] on the validity of the advanced potential and the subsequent discussions that appear to suggest that the use of the advanced potential may be untenable in relativistic electromagnetic theory [13]. Even to this date, the 1908 Ritz–Einstein disagreement continues to raise intense arguments and counter-arguments [14,15]. However, our use of the advanced potential in deriving approximate external acoustic models can be justified primarily by the observation that the classical laws of physics (to which the acoustics field belongs) discovered by Galileo, Newton and Einstein are time-symmetric. Another justification comes from the fact that our external acoustic pressure field computations involve local phenomena, not the universally global one as required in astronomy and cosmology. Third, a similar concept has been found in recent applications of the time-reverse concept in acoustics [16,17]. In other words, as acoustic signals are invariant under time-reversal, each packet of sound that comes from a source can be reflected, refracted or scattered. Consequently, a set of reflected waves can retrace all of the scattering paths, converging at the original source just as if time was going backwards. The rest of the paper is organized as follows.

Section 4 explains the computational procedure for acoustic–structural interaction. In Section 5, the modal study of the proposed model are performed for sphere and infinite cylinder. In the case of sphere and infinite cylinder, the analytic modal solutions [18] exist and through a comparison with the analytic modal solutions, the characteristics of the present model are investigated.

In Section 6, to evaluate the performance of the present model, transient analyses of submerged cylindrical shells subjected to a series of incident waves are performed. These transient responses are solved, first, by the superposition of the solutions of the modal form of the new pressure equation; and, second, by interfacing the finite element structural models with the present second-order boundary element pressure equations. Numerical performance of the proposed model is compared with the classical analytical solutions [19,20] and the Doubly Asymptotic Approximations (DAAs) [4,6]. Finally, the present model is applied to the complex ring circular cylinder for shock analysis [21].

The present parameterized model thus derived shows that: (1) the maximum convergent temporal order of the coupled acoustic pressure equation is at most two; (2) several existing approximate models fail to satisfy the initial impulse response condition, thus they may yield erroneous impulse responses that are important for inverse identification applications; (3) the present parameterized approximate model may be tailored to specific applications for problems where the computation of pressure fields radiating from the flexible surface constitutes a key interest. The present approximate model shows accurate transient responses for acoustic–structural interaction problem through the submerged spherical and cylindrical shells. In particular, it shows accurate result at early time. Finally, through the shock analysis using discrete models, the possibility for applications of our approximation model to practical engineering problems is validated.

Thus, a major contribution in the present paper is the development of a parameterized second-order external acoustic model that can be applicable to general acoustic–structural interface surfaces. The present model yields exact initial impulse responses that are important for inverse acoustic-field identification applications.

2. Pressure approximations based on Kirchhoff’s retarded potential formula

Kirchhoff’s retarded potential formula for describing the expanding or radiating waves can be expressed as [8]

\[
4\pi \varepsilon \phi(P, t) = - \int \left\{ \frac{1}{r} \frac{\partial \phi(Q,t)}{\partial n} + \frac{\partial r}{\partial n} \phi(Q,t) + \frac{1}{c^2} \frac{\partial r}{\partial t} \phi(Q,t) \right\} dS_Q, \tag{1}
\]

where \( \phi \) the retarded velocity potential, \( t \) time, \( c \) the speed of sound in fluid, \( r \) is the distance from \( P \) to a typical point \( Q \) on the surface \( S \); \( \partial /\partial n \) denotes differentiation along the outward normal to \( S \); \( \varepsilon \) is the solid angle that takes on (1, 0.5, 0) depending on whether the point \( P \) is within the acoustic domain, on the surface \( S \), or inside the enclosed surface \( S \); and, \( t_r = (t - \frac{r}{c}) \) denotes the retarded time. Substituting the Laplace transform of the retarded potential

\[
\int_0^\infty e^{-st} \phi(Q,t_r) dt = e^{-st/c} \cdot \phi(Q,s), \quad \phi(Q,s) = \int_0^\infty e^{-st} \phi(Q,t) dt \tag{2}
\]

into Kirchhoff’s retarded potential formula (1) yields the following Laplace-transformed equation:

\[
\int_S \left\{ \phi(Q,s) \frac{\partial r}{\partial n} \frac{1}{r^2} (1 + rs/c) + \frac{1}{c} \frac{\partial r}{\partial n} \phi(Q,s) \right\} e^{-st/c} dS_Q + 4\pi \varepsilon \phi(P,s) = 0, \tag{3}
\]
where the initial-value $\phi(Q, 0)$ is dropped because it represents an integral of the pressure at time $t = 0$ and vanishes for an infinitely small initial time increment, $\delta t \rightarrow 0$.

The conversion of the above equation in terms of the interface velocity of the structure normal to the surface, $u_n$, and the pressure on the surface, $p$, is obtained by the following relations:

$$ u_n = -\frac{\partial \phi}{\partial n}, \quad p = \rho \phi. \quad (4) $$

Substitutions of the Laplace-transformed forms of the preceding relations into Eq. (3) lead to the following acoustic pressure $p$, vs. the structural normal velocity ($u_n$):

$$ \int_S \tilde{p}(Q, s) \frac{\partial r}{\partial n} \left(1 + rs/c\right) e^{-\gamma/c} dS_Q + 4\pi \delta P = \int_S \frac{\partial r}{\partial n} u_n(Q, s) \frac{1}{\rho s} e^{-\gamma/c} dS_Q. \quad (5) $$

The Laplace-transformed counterparts, Eqs. (3) and (5), state that the contributions of Kirchhoff’s retarded potential formula (1) from the previous states are expressed in terms of the delay operator $e^{-\gamma/c}$. It should be noted that various approximations, both in the time and the Laplace domain methods, amount to how this delay operator is approximated. To see the impact of the delay operator on the resulting approximate models, in particular, stable or unstable models, let us examine the following expansions of $e^{-\gamma/c}$:

**Late-time unstable filter** $e^{-\gamma/c} \approx 1 - sr/c$ when $|sr/c| \ll 1$.

**Early-time stable filter** $(s \rightarrow \infty)$ $e^{-\gamma/c}$

$$ \approx \frac{1}{1 + sr/c} \quad \text{when} \quad |sr/c| \gg 1. $$

**All-frequency filter** $e^{-\gamma/c} \approx 1$ for $0 \leq |sr/c| \leq \infty$. \quad (6)

Therefore, the late-time filter is for $sr/c \ll 1$, which corresponds in time domain to $ct/r \gg 1$. On the other side, the early-time filter is for $sr/c \gg 1$ which corresponds to $ct/r \ll 1$ and has an effect on spatially local region ($ct \ll r$) [22]. In order to get physical understanding of the above approximations, we will view them as filters and plot the magnitude vs. $|sr/c|$ where the Laplace variable $s$ is replaced by $j\omega$ as shown in Fig. 1. Here, the Laplace-transformed retarded potential is the function of $j\omega r/c$ related with frequency $(\omega)$ and space $(r)$. Of the three filters, the two-term Taylor series labeled as late-time unstable filter can be easily shown to be unstable and hence is discarded.

### 2.1 Boundary integral terms employed in the present paper

In approximating the retarded potential Eq. (5) as well as its corresponding advanced potential to be introduced shortly, we will utilize the following boundary generic integral expressions:

$$ B_p(p, s) = \int_S \frac{\partial r}{\partial n} p(Q, s) dS_Q, \quad B_r p(p, s) = \int_S \frac{1}{r} \frac{\partial r}{\partial n} p(Q, s) dS_Q, $$

$$ B_q p(p, s) = \int_S \frac{1}{r^2} p(Q, s) dS_Q + 4\pi \delta P \tilde{q} p(p, s), $$

$$ A_u(p, s) = \int_S \tilde{u}(Q, s) dS_Q, \quad A_1 u(p, s) = \int_S \frac{1}{r} \tilde{u}(Q, s) dS_Q, $$

$$ A_2 u(p, s) = \int_S \frac{1}{r^2} \tilde{u}(Q, s) dS_Q. $$

In the remainder of the paper we will refer to these boundary integrals in the derivation of approximate acoustic scattering equations.

### 2.2 An early-time approximation $(e^{-\gamma/c} = 1/(1 + sr/c))$

The plane wave approximation was proposed for the early-time responses by Mindlin and Bleich [1] and Fellipa [5]. It will be shown below that the application of the present early-time stable filter listed in Eq. (6) leads to a consistent early-time approximation with curvature corrections. To this end, substitution of $e^{-\gamma/c} = 1/(1 + sr/c)$ into Eq. (5), we obtain

$$ \int_S \tilde{p}(Q, s) \frac{\partial r}{\partial n} \frac{1}{r^2} dS_Q + 4\pi \delta P \tilde{p} = \int_S \frac{\partial r}{\partial n} u_n(Q, s) \frac{1}{\rho s} \frac{s}{(1 + sr/c)} dS_Q. \quad (8) $$

Since $(sr/c \gg 1)$, the term $s/(1 + sr/c)$ in the right-hand side of the above expression is approximated as

$$ s/(1 + sr/c) \approx \frac{c}{r}. \quad (9) $$

Substituting this into Eq. (8) and after generic discretization of the resulting integral equation leads to

$$ B_q p(p, s) = \rho c A_2 \tilde{u}_n(p, s), \quad (10) $$

where $p$ and $u_n$ are the discretized pressure and structural surface velocity, respectively.

**Remark 1.** Observe that the matrix $B_q$ embodies the curvature effect, i.e., $(\frac{c}{r^2})$. Hence, we conjecture that the early-time approximation derived in Eq. (8) and its discrete version is akin to the curvature correction proposed by Fellipa [5]. The classical plane wave approximation may be realized by setting

$$ \frac{\partial r}{\partial n} \rightarrow 1 $$

so that Eq. (10) reduces to

$$ \tilde{B}_q \tilde{p}(p, s) = \rho c A_2 \tilde{u}_n(p, s) \downarrow $$

with $\{ \epsilon = 0. B_q = A_1 \} \Rightarrow \rho = \rho c u_n$. \quad (12)

Thus, from the view point of the present derivation as seen from the above equation, the plane wave approximation given by $(p = \rho c u_n)$ for the modeling of early-time responses omits the source-point singular term, $4\pi \delta P \tilde{Q} p$.

### 2.3 All-frequency approximation $(e^{-\gamma/c} = 1)$

Substituting $(e^{-\gamma/c} = 1)$ into the pressure Eq. (5) we obtain

![Frequency Characteristic of Delay Operator Approximations](image-url)
\[ \int \frac{\partial r}{\partial t} \frac{1}{r^2} (-\nabla \cdot \nabla \psi) dS + \frac{4\pi \epsilon\rho}{c} = \int \frac{\partial u_r}{\partial t} \frac{1}{r} \rho \delta dS \]

\[ \{ sB_1 + cB_2 \} \bar{p}(P, s) = \rho c s A_1 \bar{u}_n(P, s). \]

When one invokes the plain wave realization (11) to the above equation, the resulting approximation leads to

\[ \frac{\partial r}{\partial t} \rightarrow 1, \quad B_1 = A_1 \quad \Rightarrow \quad \{ sA_1 + cB_2 \} \bar{p}(P, s) = \rho c s A_1 \bar{u}_n(P, s). \]

The last equation in the above derivations obtained from the application of the all-frequency filter \( e^{-\omega t/c} = 1 \) is in fact identical to the first-order Doubly Asymptotic Approximation (DAA1) derived by Geers [22] by the asymptotic matching of his early-time approximation and late-time approximation.

As DAA1 has been widely used in underwater shock analysis [9], we illustrate a converse process of the doubly asymptotic approximation for the derivation of the DAA1. In Eq. (14), we consider the two cases:

For the early-time asymptote, we have

\[ \lim_{s \to 0} \left\{ \{ A_1 + \frac{1}{s} cB_2 \} \bar{p}(P, s) - \rho c A_1 \bar{u}_n(P, s) = 0 \right\} \]

\[ \bar{p}(P, s) = \rho c \bar{u}_n(P, s). \]

In order to effect a late-time asymptote, we replace \( s\bar{u}_n \) by its acceleration counterpart, viz.,

\[ a_n(t) = \frac{d}{dt} u_n(t) \quad \Rightarrow \quad \bar{u}_n(s) = s\bar{u}_n(s) \]

(16)

to arrive at

\[ \lim_{s \to 0} \{ sA_1 + cB_2 \} \bar{p}(P, s) - \rho c A_1 \bar{a}_n(P, s) = 0 \}

(17)

\[ \bar{p}(P, s) = M_n \bar{a}_n(P, s), \quad M_n = \rho A_2^{-1} A_1, \]

where \( M_n \) [24] can be considered an added mass per unit surface area. This means that the pressure at late-time period is dominated by the added mass inertia.

In order to derive the DAA2, Geers assumed the following transfer function:

\[ \frac{p(s)}{u_n(s)} = \frac{N(s)}{D(s)}, \]

(18)

whose asymptotes \( s \to \infty, s \to 0 \) will satisfy the early-time approximation (15) and the late-time approximation (17), respectively; The transfer function (18) thus determined yields the same approximation as derived by invoking the all-frequency filter (14).

Comparing the present derivation of approximate acoustic equation from the retarded potential by approximating the delay exponential by approximate stable filters vs. the derivation of the DAA1 by doubly asymptotic matching, we observe at least one encouraging conjecture. That is, as there are almost an unlimited number of filters that can approximate the delay exponential, one may hope to derive a family of approximate stable pressure equations that can bypass further asymptotic matching, especially for higher-order approximate models.

This has motivated the present authors to develop a set of second-order approximate acoustic models that are more accurate than the DAA1 and perhaps consistent compared with the DAA2. The motivation for the development of the DAA2 was to improve the accuracy as the DAA1 has been known to cause excessive pressure attenuation. In fact, Geers and co-workers [4] proposed a set of second-order approximations labeled as DAA2s. Of these DAA2 variations, the modal DAA2 specialized for the spheres is by far the best approximation [22]. To the best knowledge of the present authors, a discrete or matrix form that corresponds to the modal spherical DAA2 is not available in the open literature.

This has motivated us to develop a second-order approximation as detailed in the following section. It should be emphasized that our approach is to utilize the salient properties of filter operators, thus alleviating consistency issues that may arise in any asymptotic matching endeavors.

### 2.4. Evaluation of retarded potential-based approximate models

As alluded to in Section 1, the objectives of the present work are to improve accuracy of approximate acoustic models for external acoustic–flexible structure interactions. In particular, we would like to develop an accurate model that can be used for design optimization, which can incorporate experimentally identified pressure characteristics. Since system identification procedures almost exclusively utilize impulse responses and/or frequency response functions, the approximate models we intend to develop must possess the required impulse–response consistency.

Second, the model we intend to propose should be applicable for general structural geometries, not just spheres and cylinders. This means modal models developed for spheres and cylinders are often not extendable to general geometries, hence the preferred approximate models should lend naturally to a computationally implementable discrete form.

#### 2.4.1. Early-time consistency requirement

To address the first issue, viz., the requirement of correct identification of impulse response functions, we summarize exact modal equation obtained by solving the wave equation [18] and the three DAAs below as specialized to an elastic sphere.

**Exact modal equation:**

\[ \frac{\hat{p}(s)}{u_n(s)} = \frac{K_n}{K_n} = \begin{cases} \frac{s}{(s^2 + s) / (s^2 + 2s + 2^n)}, & \text{for } n = 0, \\ \frac{s}{s^2 + (1 + n) s + (1 + n)^2}, & \text{for } n = 1, \\ \frac{s}{s^2 + (1 + n) s + n(1 + n)^2}, & \text{for } n = 2, \end{cases} \]

\[ \text{DAA1} : [s + (1 + n) ] p_n = s u_n, \quad (20) \]

**DAA2** (1978):

\[ [s^2 + (1 + n) s + (1 + n)^2] p_n = [s^2 + (1 + n) s + n(1 + n)^2] u_n, \quad (21) \]

**DAA2** (1994):

\[ [s^2 + (1 + n) s + n(1 + n)] p_n = [s^2 + ns] u_n, \quad (22) \]

It should be noted that there is no known discrete counterpart of the DAA2 labeled as DAA2 (1994) [23]. Applying the initial-value theorem to the above Laplace-transformed equations, we obtain the temporal magnitude of the impedance at time \( t = 0 \) when unit impulse velocity is applied to the elastic sphere as listed below

\[ \lim_{t \to 0} \left[ \frac{p_n(t)}{u_n(t)} \right] = \delta (0) - 1, \quad \text{for exact modal equation}, \]

\[ \lim_{t \to 0} \left[ \frac{p_n(t)}{u_n(t)} \right]_{\text{DAA1}} = \delta (0) - (n + 1), \quad \text{for DAA1}, \]

\[ \lim_{t \to 0} \left[ \frac{p_n(t)}{u_n(t)} \right]_{\text{DAA2} (1978)} = \delta (0) - 1, \quad \text{for modal form of DAA2 (1978)}, \]

\[ \lim_{t \to 0} \left[ \frac{p_n(t)}{u_n(t)} \right]_{\text{DAA2} (1994)} = \delta (0) - 1, \quad \text{for modal form of DAA2 (1994)}, \]

where \( n \) is the modal number when the solution for both pressure \( (p(t)) \) and velocity \( (u(t)) \) are expanded in terms of both pressure harmonics.

It is noted that the DAA1 satisfies the impulse impedance only for \( n = 0 \) mode; the discrete DAA2 [4] does not satisfy the initial impulse impedance. The spherical modal form of DAA2 [23] does satisfy the initial impulse impedance. However, its generalization to general geometries has not been presented to date.
2.4.2. Unstable filters due to time delay exponential for second-order models

It is shown in the previous section that the pressure approximations that are obtained by approximating Kirchhoff’s retarded potential formula are stable when employing the all-frequency filter \( \exp(-st/\varepsilon) \approx 1 \) that produces the first-order DAA, and the early-time filter \( \exp(-st/\varepsilon) = 1/(1 + s/\varepsilon) \) that produces a zero-order early-time approximation. However, the late-time approximation employing \( \exp(-st/\varepsilon) \approx 1 - s/\varepsilon \) is an unstable approximation, primarily due to the inherent destabilizing delay exponential \( \exp(-st/\varepsilon) \), a well-known fact in delayed feedback theory.

A general consistent form of filters emanating from the delay exponential \( \exp(-st/\varepsilon) \) for second and beyond orders, e.g.,

\[
e^{-st/\varepsilon} = \frac{1}{1 + s/\varepsilon} + \frac{1 + \varepsilon s^2}{1 + s/\varepsilon^2} + \cdots
\]

(24)
can be shown to result in computationally unstable pressure approximation. For example, consider the so-called \((2-2)\) Padé approximation of \( \exp(-st/\varepsilon) \):

\[
e^{-st/\varepsilon} = \frac{1 - \varepsilon s^2/2c + \varepsilon s^2/2c^2}{1 + s/2c + \varepsilon s^2/2c^2} = 1 - \frac{\varepsilon s^2/2c}{1 + s/2c + \varepsilon s^2/2c^2}
\]

(25)

This filter has unit magnitude for all the frequency ranges just as the all-frequency filter. However, it is not implementable as an ordinary differential equation when the corresponding approximation is transformed back to its time-domain equation. In other words, a filter of rational fraction nature does not lend to a computationally tractable equation.

We now present new parameterized second-order approximate external acoustic models that are early-time consistent impedance, stable and computationally implementable upon spatial discretization.

3. Proposed parameterized model

In this section we present a hybrid potential, a potential that combines retarded and advanced potentials for the derivation of higher-order external acoustic model equations. In doing so, we show that the maximum limit of order in time is two, and subsequent high-order approximations are not spatially convergent. We then evaluate the resulting parameterized second-order models for model fidelity. Once the model parameter is determined as compared with the analytical solution of wave equation for an elastic sphere, the parameter is then transformed into discrete matrix so that the spatially discretized model equation can be applicable to general surface geometries.

3.1. Combined use of retarded potential and advanced potential

The inherent destabilizing property of the time delay exponential associated with the retarded potential for higher-order approximation can be obviated by employing the advanced acoustic potential defined as

\[
\phi_s = \phi(Q, t_s) = \phi(Q, t + \frac{c}{\varepsilon}), \quad \int_0^\infty e^{-\varepsilon t}s \phi(Q, t_s) dt = e^{\varepsilon t_s} \phi(Q, s).
\]

(26)

When one combines the retarded and advanced potentials in accordance with the weighting rule stipulated in [12], the following weighted velocity potential results:

Present modified potential:

\[
\phi_{mod} = \frac{1}{2}(1 - \varepsilon) \phi_s + \frac{1}{2}(1 + \varepsilon) \phi_a,
\]

(27)

Retarded potential: \( \phi_s = \phi(Q, t - r/c) \),

(28)

Advanced potential: \( \phi_a = \phi(Q, t + r/c) \),

(29)

whose Laplace-transformed expression is given by

\[
\tilde{\phi}_{mod} = \left\{ \frac{1}{2}(1 - \varepsilon) \phi_s + \frac{1}{2}(1 + \varepsilon) \phi_a \right\} \phi(Q, s),
\]

(30)

where \( \varepsilon \) is a parameter to be determined.

Two-term Taylor expansion of \( \tilde{\phi}_{mod} \) gives

\[
\tilde{\phi}_{mod} \approx (1 + \varepsilon s/\varepsilon) \tilde{\phi}(Q, s), \quad \varepsilon > 0 \quad \text{for stability},
\]

(31)

which is valid for late time, i.e., \( (sr/c) \ll 1, \quad s \rightarrow 0 \).

When the approximate modified potential \( \tilde{\phi}_{mod} \) is used in Eq. (5) in place of \( \phi_s \), we obtain

\[
\int_s \rho \tilde{u}(Q, s) \frac{\partial r}{\partial n} \frac{1}{v_1} (1 + sr/c) (1 + \varepsilon s/\varepsilon) dS_0 + 4\pi \varepsilon \tilde{e} \tilde{p}(P, s) \approx \rho \int_s \tilde{u}(Q, s) \frac{1}{v_1} dS_0,
\]

(32)

where the pressure and the velocity relations given in Eq. (4) is utilized as before.

To gain insight into the characteristics of the above second-order equation, we rearrange it to the following form:

\[
\left\{ \int_s \tilde{p}(Q, s) \frac{\partial r}{\partial n} \frac{1}{v_1} (1 + sr/c) dS_0 + 4\pi \varepsilon \tilde{e} \tilde{p}(P, s) - s\rho \delta_0 \int_s \tilde{u}(Q, s) \frac{1}{v_1} dS_0 \right\} + \chi \left\{ \int_s \tilde{p}(Q, s) \frac{\partial r}{\partial n} \frac{1}{v_1} (1 + sr/c) \frac{c}{\varepsilon} dS_0 - \frac{s^2 \rho}{c} \delta_0 \int_s \tilde{u}(Q, s) dS_0 \right\} = 0,
\]

(33)

whose generic discrete version can be expressed by using Eq. (7) as

\[
\left\{ \left( \frac{s^2}{c^2} B_1 + \frac{s}{c} B_2 \right) \tilde{p} - s\rho \delta_0 \tilde{A}_u \right\} + \chi \left\{ \left( \frac{s^2}{c^2} B_1 + \frac{s}{c} B_2 \right) \tilde{p} - \frac{s^2 \rho}{c} \delta_0 \tilde{A}_u \right\} = 0,
\]

(34)

where the weighting parameter, \( \chi \), for the continuum case is now generalized to its matrix counterpart for the discrete equation, \( \chi \).

It is observed that the expressions in the first and second braces in Eqs. (14) and (34) correspond to the all-frequency approximation given by Eq. (13) and the additional terms due to the advanced potential, viz., \( \chi = 1 \), respectively.

3.2. Maximum convergent order of present approximations is two

It is tempting to expand the hybrid potential \( \phi_{mod} \) beyond the first-order terms, i.e.,

\[
\phi_{mod} \approx \left( 1 + \varepsilon s/\varepsilon \right) \phi(Q, s), \quad \varepsilon \geq 0 \quad \text{for stability},
\]

(35)

so that one obtains the following third-order equations:

\[
\int_s \tilde{p}(Q, s) \frac{\partial r}{\partial n} \frac{1}{v_1} (1 + sr/c) \left\{ 1 + \varepsilon s/\varepsilon + \frac{1}{2} s^2 r^2/c^2 \right\} dS_0 + 4\pi \varepsilon \tilde{e} \tilde{p}(P, s) \approx \rho \int_s \tilde{u}(Q, s) \frac{1}{v_1} dS_0,
\]

(36)

which is rearranged as

\[
\left\{ \int_s \tilde{p}(Q, s) \frac{\partial r}{\partial n} \frac{1}{v_1} (1 + sr/c) dS_0 + 4\pi \varepsilon \tilde{e} \tilde{p}(P, s) - s\rho \delta_0 \int_s \tilde{u}(Q, s) \frac{1}{v_1} dS_0 \right\} + \chi \left\{ \int_s \tilde{p}(Q, s) \frac{\partial r}{\partial n} \frac{1}{v_1} (1 + sr/c) \frac{c}{\varepsilon} dS_0 - \frac{s^2 \rho}{c} \delta_0 \int_s \tilde{u}(Q, s) dS_0 \right\} + \frac{1}{2c^2} \left\{ \int_s \tilde{p}(Q, s) \frac{\partial r}{\partial n} (s^2 + s^2 r^2/c) dS_0 - \rho \int_s \tilde{u}(Q, s) s^3 r dS_0 \right\} = 0.
\]

(37)

Notice that the first two lines in the above equation is the same as the approximate model derived in Eq. (33) that is obtained by the
two-term expansion of the proposed combined potential ($\phi_{\text{mod}}$). The terms in the third line represents the contribution of the second-order terms in the expansion of $\phi_{\text{mod}}$ in Eq. (35). The integral terms of the form $\int f_j \bar{u}(Q, s) r dS, \int p_j (Q, s) r dS$ therein become divergent as $r \to \infty$. In other words, they are not admissible. This implies that, from the context of the present formulation, the maximum temporal order of convergent interaction models is two.

3.3. Plane wave modification

It should be noted that the present second-order model (34) is, strictly speaking, valid for late time approximation as the expansion given in Eq. (31) is for $(sr/c < 1 \to s \to 0)$. Hence, while it does not require an asymptotic matching of early-time and late-time approximations, as was necessary in the derivation of the DAAs, the basic second-order external acoustic–structure interaction model derived in Eq. (34) needs two modifications: plane wave approximation and a parameterized representation of the weighting factor $\chi$ and subsequently the weighting matrix $X$. We present plain wave modification below.

The approximate parameterized model for external acoustic field interacting with flexible structures derived in Eq. (34) has been obtained by expanding the delay and advance exponential to their first order. This implies, by virtue of the initial $(s \to \infty)$ and final $(s \to 0)$ value theorems of the Laplace transform, that the approximate model thus derived would offer higher model fidelity for the late-time response than for early-time response. This means that among the five coefficient operators ($B_1, B_2, B_3, A_1, A_2$) in the approximate model (34), the two zeroth-order terms ($B_1, A_1$) should need no further modifications. This leaves the two remaining operators, viz., ($B_2, B_3$), as modification candidates in order to improve the model fidelity for the early-time responses.

The plane wave approximation introduced in Eq. (11) implies that initial wave arrays are scattering from the normal of the surface. In other words, in the initial stages of the pressure transients the forces acting on the structural surface are computed as if the surface is flattened as a plane on which $B_2$ and $B_3$ are modified as

$$B_2(P, s)|_{s=1} = A_1(P, s), \quad B_3(P, s)|_{s=1} = A_2(P, s).$$

This is because, in physical terms, the direction of the wave path and the normal to the interaction surface remain parallel for plane waves. With the above modifications, the parameterized second-order external acoustic model (34) becomes

$$s^2X\mathbf{u}(P, s) + \text{sc}(I + X)(A_1\mathbf{p}(P, s) + c^2 B_2 \mathbf{p}(P, s)) = s^2pcX\mathbf{u}(P, s) + s\text{pc} A_2 \mathbf{u}(P, s).$$

(39)

It is once again noted that the present parameterized model (39) has not resorted to asymptotic matching as was the case for the DAAs.

3.4. Determination of the parameterized weighting matrix ($X$)

The determination of the advanced potential weighting parameter, $\chi$ for the continuum or modal equation and $X$ for the discrete equation can be viewed as a compensation for the gross approximation committed in the plane wave approximation introduced in Eq. (39). This is because the weighting parameter shows up only for the three terms affected by the plane wave approximation. To this end, we introduce the analytic modal specific acoustic impedance for an elastic sphere and, by comparing the analytical specific acoustic impedance to that of the present model equation, we determine mode-by-mode weighting parameters. Subsequently, we generalize the mode-by-mode weighting parameters to a single discrete weighting matrix.

3.4.1. Determination of mode-by-mode weighting parameters, $\mathbf{x}_n$

In the parameterization search, we have been guided by the exact mode-by-mode wave solution’s roots for a sphere [18].

The exact solution for spherical wave equation is expressed as

$$\frac{\partial u_n(s)}{\partial s} \mid_{\text{exact}} = -\frac{k_n}{k_n'},$$

(40)

where $k_n(s)$ is the nth order modified spherical Bessel function of the third kind given as

$$k_n(z) = \frac{\pi}{2} z^{\frac{n}{2}} \sum_{m=1}^{\infty} \Gamma_{mn} z^{-(m+1)}, \quad \Gamma_{mn} = (n + m)! /[2^m m!(n - m)!],$$

(41)

In the above equation, $u_n$ is the non-dimensionalized radial velocity of the spherical shell; $p_n$ is the non-dimensionalized pressure; $p$ and $c$ are the density and the speed of sound in the acoustic medium. Also, in the above equation the time $t$, the velocity $u_n$, and the pressure $p$ are non-dimensionalized via $t = t/c, u_n = w/a, R = r/a$ and $p = p - p_c/c^2$ where $a$ is the radius of a sphere shown in Fig. 5. The exact solution for a spherical wave equation will be explained in detail later.

For comparison purposes, the mode-by-mode form of the present second-order approximate Eq. (39) for the case of elastic spheres can be shown to read

$$\frac{p_n}{u_n} = \frac{s}{X_n s^2 + (1 + X_n s) s u_n + (1 + n) s^2 p_n} = X_n s^2 u_n + s u_n$$

(42)

$$\left[\frac{p_n}{u_n}\right]_{\text{present}} = s/\left(s^2 + \left(1 + X_n s\right) s + (1 + n) s^2 p_n\right),$$

where $X_n$ denotes the mode-by-mode parametrization of the weighting matrix $X$.

From the analytic mode-by-mode impedance that relates the pressure to the structural velocity given in Eq. (19), one finds that there are $(n + 1)$-poles and $(n + 1)$-zeros for each mode $n$. On the other hand, from the modal form of the present model Eq. (42), for $n > 1$ there are only two poles, and one zero at the origin and the other zero along the negative $s$-axis, provided $X_n > 0$. First, observe that the present approximate modal model (42) exactly matches that of the analytical impedance for $n = 0$ and $n = 1$ if one chooses $X_n = 1$.

Second, the temporal magnitude of the impedance at time $(t = 0)$ when unit impulse velocity is applied to the elastic sphere for the present modal form is evaluated as

$$\lim_{t \to 0} \left[\frac{p_n(t)}{u_n(t)}\right]_{\text{present}} = \delta(0) - 1,$$

(43)

which agrees with the case of the analytical case (23) regardless of the weighting parameter $X_n$. This means that, regardless of the values assigned to $X_n$, the present approximate model satisfies the important desirable impedance property. Therefore, we have a complete freedom to choose the modal weighting parameters matrix, $X_n$, consequently the discrete weighting matrix $X$.

Third, for $n = 0, 1$ if we select $X_n = 1 + \varepsilon$, the corresponding modal impedance becomes

$$\left[\frac{p_n}{u_n}\right]_{\text{approx}} = \begin{cases} \frac{\frac{\varepsilon^2 + 1}{\varepsilon^2 + 2} + (1 + 1/\varepsilon^2)}{\varepsilon^2 + 2} & \varepsilon - \varepsilon_0 = s/(s + 1), \quad n = 0, \\ \frac{(1 + X_n s)/\varepsilon^2}{1 + (1 + n)/\varepsilon^2} & \varepsilon - \varepsilon_0 = \frac{1 + X_n s}{\varepsilon^2 + 2}, \quad n = 1, \end{cases}$$

(44)

which reproduces the analytical modal impedance (19) exactly.

Fig. 2 shows the mode-by-mode poles of the analytic pressure characteristic equation given by Eq. (19). Also plotted are the characteristic roots of the present parameterized modal Eq. (42).
Notice that, for an even \( n \), the analytical impedance has one negative real roots, and the rest \( (n - 1) > 0 \) manifest as \((n/2)\)-complex pairs; and, for an odd \( n \), they form \((n + 1)/2\)-complex pairs. On the other hand, for all \( n \), the present parameterized modal equation possesses only one pair of complex roots. Hence, a direct mode-to-mode comparison of the characteristic roots of the analytical case to the present second-order approximate model is not possible, except for \( n = 0 \) and \( n = 1 \). It is this inadequacy of the present approximate model that limits a tailoring ability of the present model to faithfully capture the analytical characteristic roots.

In view of such limitations, we have endeavored to capture the oscillatory part of the present second-order model to closely match the imaginary part of the dominant complex roots of the analytical characteristic roots, and let the negative real part represent an average decay rate of the analytical roots. This is illustrated in Fig. 2. Also shown in that figure is the root sensitivities of the present model for three cases of \( \chi_a \):

\[
\chi_a^{-1} = d n. \quad d = (0.75, 1.0, 1.25), \quad \{n = n, n > 0\}
\]

\[
\chi_a^{-1} = \tilde{n}, \quad \{\tilde{n} = 1, \ n = 0\}.
\]

(45)

Clearly, it is seen from the magnified part of Fig. 2 that the smaller the constant \( d \) is, the smaller the damping. However, the oscillatory part also decreases.

Fig. 3 shows the approximate roots for the case of \( d = 1 \) vs. the analytical characteristic roots. Observe that the present approximate model captures the oscillatory part reasonably well while the decaying part to be at most an average of the real parts of the analytical roots.

![Fig. 2. Mode-by-mode characteristic roots of the pressure equation for a spherical shell geometry.](image1)

![Fig. 3. Mode-by-mode characteristic roots of the pressure equation for a spherical shell geometry – continued (weighting parameter chosen: \( \chi_a^{-1} = n \)).](image2)
Based on the preceding discussions, we propose the following mode-by-mode parameterization in the form of:

$$\chi_n^{-1} = \begin{cases} \hat{n}, & \text{when } \{n = 1, n = 0\} \\ b_0 + b_1 n + b_2 n^2, & \{n = 1, n = 1\} \end{cases}$$  \hspace{1cm} (46)$$

In the above modal parameter coefficients, $b_0$ infuses mass-proportional damping, $b_1$ introduces viscous damping and $b_2$ plays the role of stiffness proportional damping.

### 3.4.2. Determination of discrete weighting parameterization matrix, $X$

While it is relatively straightforward to determine the above mode-by-mode parameter, its translation into a discrete matrix form has been a challenge and can in no way, be uniquely determined. After a series of trial-and-error attempts, we have selected the following discrete parameterization matrix:

$$C = X^{-1} = b_1 N^{-1} B_2 - (b_1 - b_0) I + S,$$

$$N = A_1 \bar{A}_1^{-1},$$  \hspace{1cm} (47)$$

$$X = \left[ \int_S \frac{1}{r} \, ds \right]^{-1} \left[ \int_S \frac{1}{r} \frac{\partial r}{\partial n} \, ds \right]^{-1}$$

where $S = 2B_1 A_1^{-1}$ is a stabilization matrix to be explained shortly and $B_2$ is a matrix which all elements of each column corresponding to $P$ point consist of $\frac{1}{2} \frac{1}{n} dS$, where $S$ is a stabilization matrix to be explained shortly.

To see the critical role of the stabilization matrix $S$ in the above parameterization matrix obtained, the spherical shell geometry is discretized with 385 4-noded elements, and the first six eigenvalues of $(N^{-1} B_2 - I)$ and $S$ are computed as shown in Fig. 4. Note that, without the stabilization term $(S)$, the parameterization matrix would have one negative root corresponding to $n = 0$, which can cause instability. However, with the stabilization matrix the eigenvalues of $X^{-1}$ shifts the negative $(n = 0)$-root near to the $(n = 1)$-root, thus stabilizing the $(n = 0)$-root, while leaving the rest of the roots intact. In other words, the stabilization matrix $S$ shifts the $(n = 0)$-pole away from the negative zone to the stable positive value, making $X \approx I$. For the rest of the roots, as already mentioned, the stabilization matrix hardly changes their values.

Substituting Eqs. (47) into (39), we obtain the discrete second-order external acoustic model that interacts with flexible structures:

$$\mathbf{A} \mathbf{p} + c(1 + C A_1 \mathbf{p}) + c^2 C B_1 \mathbf{p} = \rho c \mathbf{u} + \rho c^2 C A_1 \mathbf{u},$$  \hspace{1cm} (48)$$

where $C$ is given by Eq. (47). Therefore, the present approximation can be applied to general engineering problem. But the present approximation is restricted to convex surface since in the case of concave surface, reflect waves from the other surface happen.

We note that the above present discrete pressure model is specialized, for an elastic spherical structure, to the following non-dimensionalized equation in terms of the discrete version of the Legendre functions, $\psi_n$,

$$\psi_n \psi_n = \frac{4\pi}{2n + 1},$$

$$\psi_n B_2 \psi_n = \frac{4\pi(n + 1)}{2n + 1},$$

$$\psi_n S \psi_n = \delta(0), \quad \delta(0) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0 \end{cases}$$

$$\chi_n^{-1} = \psi_n C \psi_n = (n + 1) - 1 + \delta(0)$$

$$\mathbf{p}_n + [n + 1 + \delta(0)] \mathbf{p}_n + [(n + 1) + \delta(0)] \mathbf{p}_n = \mathbf{u}_n + [n + \delta(0)] \mathbf{u}_n.$$  \hspace{1cm} (49)$$

The present discrete equation using the weighting parameter, (47) shows similar modal equation, (49) to the curvature corrected DAA$_2$ [23] without using the stabilization matrix $(S)$. In this case, the curvature corrected DAA$_2$ can be unstable as shown in Fig. 4 and the detailed discussion is expressed in next section.

### 3.4.3. Comparison of the present pressure model with the DAA models

The second-order DAA presented in the seminal works of Geers [4], Felippa [5] and Geers and Felippa [6] may be expressed, in conformity with the present boundary integral expressions, as

$$\mathbf{A} \mathbf{p} + \rho c (\mathbf{A}^+_1 \mathbf{p}) + \rho c^2 (\mathbf{A}^+_1 \Omega \mathbf{p}) = \rho c \mathbf{u} + \rho c^2 (\mathbf{A}^+_1 \Omega \mathbf{u}),$$

$$\mathbf{M}_f = \frac{\rho}{2} \mathbf{B}_2^{-1} \mathbf{A}_1 \mathbf{A} + \mathbf{A}_1 \mathbf{B}_2^{-1},$$

$$\Omega = \rho c \mathbf{A}^+_1 - c \mathbf{k}_f,$$  \hspace{1cm} (50)$$

where $k$ is the curvature of each boundary element with its unit normals chosen toward the exterior of the structure.

A direct comparison of the present model (48) with the discrete DAA$_2$ (50) is not possible, except for the case of the spherical geometry. For an elastic sphere of the radius $a$, we have the following spherical modal properties:

$$\psi_n^T A \psi_n = \frac{4\pi a^2}{2n + 1},$$

$$\psi_n^T A_1 \psi_n = \frac{4\pi a^2}{2n + 1},$$

$$\psi_n^T M \psi_n = \rho \left( \frac{4\pi a^2}{2n + 1} \right) (n + 1),$$

$$\psi_n^T \Omega \psi_n = \frac{\zeta}{a} (n + 1) - \alpha \kappa a.$$  \hspace{1cm} (51)$$

Using these relations, the discrete DAA$_2$ model (50) can be transformed into the spherical, non-dimensionalized modal equation

$$\mathbf{p}_n + [(n + 1) + \kappa a] \mathbf{p}_n + [(n + 1) - \kappa a] \mathbf{p}_n = \mathbf{u}_n + [(n + 1) - \kappa a] \mathbf{u}_n,$$  \hspace{1cm} (52)$$

where $\kappa a$ is the modal curvura which, for a sphere of radius $a$ and in exact computation, becomes $\kappa a = 1/a$, and $\mathbf{p}_n$ is the modal, non-dimensionalized pressure.

We now offer the following comments regarding the discrete DAA$_2$ (50) and its spherical modal form (52). First, computer implementation of the term $\Omega$ is problematic at best if not impossible. For example, when the structural surface is discretized in terms of four-node bilinear elements or eight-node brick elements, the
curvature term will approach to zero, viz., $\kappa \to 0$. This leads to the DAA2 presented in 1978 [4], which is inconsistent for capturing the initial impulse response as shown in Eq. (23). An alternative is to use the exact interaction surface geometry to generate the curvature matrix, which would be prone to errors except for the case of spherical and cylindrical surfaces, because there will be in general two curvatures from which one may compute a principal curvature.

Second, even for a spherical idealization with padded fluid zones, computations of $\mathbf{M}_f$ would engender errors. This means that the modal term $[(n + 1) - AK_n]$ with $K_n = 1/a$ would be modified in practice by

$$[(n + 1) - AK_n] \leadsto (1 + \epsilon)(n + 1) - 1 = (1 + \epsilon)n + \epsilon, \quad |\epsilon| \ll 1$$

so that the spherical modal Eq. (52) for $n = 0$ becomes

$$\mathbf{p}_0 + (1 + \epsilon)\mathbf{p}_0 = \mathbf{u}_0 + \epsilon\mathbf{u}_0.$$  \hspace{1cm} (53)

Hence, the $(n = 0)$-modal equation can become unstable if $\epsilon < 0$.

The preceding examination implies that the curvature corrected DAA2 may not be robust for applications to general structural geometries.

Remark. The DAA2 with a curvature correction is analogous to the DAA2 with a curvature correction can be interpreted as the transformation matrix from structural and acoustic coordinate system, the pressure and displacement on the surface of a sphere can be expanded in terms of Legendre polynomial series. For example, Huang [19], Zhang and Geers [25], and other researchers obtained the analytic modal solutions for the interaction problems of a submerged elastic spherical shell excited by incident plane step wave excitation.

Even though $\mathbf{C}$ is easily implementable, it can lead to instability just as the DAA2 with a curvature correction can be unstable as examined above.

We note that the present parameterized discrete second-order model for the external pressure field is applicable to general structural surface geometries. The numerical evaluation of the present discrete parameterized acoustic interaction model (48) is presented in the next sections.

4. Acoustic–structure interaction equations

The finite element structural dynamic equations of motion are expressed as

$$\mathbf{M}_s \dot{\mathbf{x}}_s + \mathbf{K}_s \mathbf{x}_s = \mathbf{f}_s - \mathbf{G}\mathbf{p}.$$  \hspace{1cm} (56)

where $(\mathbf{M}_s, \mathbf{K}_s)$ are the structural mass and stiffness matrices, respectively; $\mathbf{x}_s$ is the structural displacement; $\mathbf{f}_s$ is the external force acting on the structure; and, $\mathbf{G}$ is a Boolean matrix that extracts the normal pressure component on the structural and acoustic interface surface.

When the structure is placed in an acoustic medium, the structural Eq. (56) is subjected to the incident and scattering pressures whose governing equation is modeled by the present external acoustic model Eq. (48). As the pressure $\mathbf{p}$ consists of the sum of the incident pressure, $\mathbf{p}'$ and scattering pressure $\mathbf{p}^s$, we decompose the pressure according to

$$\mathbf{p} = \mathbf{p}' + \mathbf{p}^s$$  \hspace{1cm} (57)

and seek the solution of the scattering pressure $\mathbf{p}^s$ as the incident pressure field is known.

On the surface of the structure submerged in an acoustic medium, the geometric compatibility leads to the following relation:

$$\mathbf{G}^T \dot{\mathbf{x}}_s = \mathbf{u}_n^i + \mathbf{u}_n^s,$$  \hspace{1cm} (58)

where $\mathbf{u}_n^i$ and $\mathbf{u}_n^s$ are the normal direction velocities of incident waves and scattering waves on the surface of structures, respectively; $\mathbf{G}^T$ is now interpreted as the transformation matrix from structural mesh to those wet-surface mesh of acoustic fluid. Therefore, both of structural and acoustic equations are coupled with each other and should be simultaneously calculated to obtain transient responses.

Using Eqs. (57) and (58), computer implementation of the coupled external acoustic–structural interaction equations are carried out by the following equations:

$$\begin{align*}
\mathbf{M}_s \dot{\mathbf{x}}_s + \mathbf{K}_s \mathbf{x}_s &= -\mathbf{G}(\mathbf{p}' + \mathbf{p}^s), \\
\mathbf{X}\mathbf{p}' + (I + \mathbf{X})\mathbf{A}_i \mathbf{p}^s + c^2 \mathbf{B}_i \mathbf{p}^s &= \rho \mathbf{C}_1 \mathbf{A}_i (\mathbf{G}^T \dot{\mathbf{x}}_s - \mathbf{u}_n^i) + \rho c^2 \mathbf{A}_i (\mathbf{G}^T \dot{\mathbf{x}}_s - \mathbf{u}_n^s), \quad \mathbf{p} = \int_0^t \mathbf{p}(t)dt.
\end{align*}$$

where $\mathbf{A}_i$ is the elemental area matrix; and we have dropped the superscript $(s)$ from the scattering pressure $\mathbf{p}^s$ for notational simplicity.

5. Modal study of the proposed model

Analytic solutions of the wave propagation equation exist for plate, sphere, and infinite cylinder. By invoking the spherical coordinate system, the pressure and displacement on the surface of a sphere can be expanded in terms of Legendre polynomial series. For example, Huang [19], Zhang and Geers [25], and other researchers obtained the analytic modal solutions for the interaction problems of a submerged elastic spherical shell excited by incident plane step wave excitation.

For comparison purposes, the mode-by-mode equations of the present pressure interaction equation is obtained in terms of Legendre polynomial series for a spherical shell. Using the resulting modal equations of the present approximate model, the mode-by-mode acoustic impedance, the poles of its impedance and acoustic–structure coupled characteristic equations can be computed. The characteristic poles of the proposed model thus obtained are then compared to those of the analytic modal solutions [18] and of the DAA2 (1978) model [4]. We will then assess the performance of the proposed model with the chosen weighting parameter (47) for the case of an infinite elastic cylinder subjected to external incident waves.

5.1. An elastic spherical shell surrounded by acoustic medium

Fig. 5 shows a flexible elastic spherical shell of radius $a$, thickness $h$, an isotropic material with Young’s Modulus $E$, density $\rho_v$, and Poisson’s ratio $\nu$. The shell thickness-to-radius ratio $h/a$ is small enough to apply thin shell theory and the longitudinal wave speed of the shell is denoted by $c_0 = \sqrt{E/\rho_v(1 - \nu^2)}$. 

![Fig. 5. A submerged spherical shell excited by a cosine-type impulsive pressure.](image-url)
The shell geometry is described using a spherical coordinate \((R, \theta, \phi)\) with its origin at 0 and an in vacuo condition of its interior. The radial and meridional displacements of the shell are denoted by \(w(\theta, t)\) and \(v(\phi, t)\), respectively. For subsequent analysis, dimensionless variables are introduced: time \((\tilde{t} = t/c_0)\), pressure \((p = p/c^2)\) and length \((w = w'/a, v = v'/a)\) where \((w, v)\) are the normal and tangential displacement, respectively. For numerical computations, the submerged spherical shell as shown in Fig. 5 has the parameters: \(h/a = 0.01, \rho_s/\rho = 7.7\) and \(c_s/c = (13.8)^{1/2}\).

### 5.2. Exact modal equation for a sphere

The acoustic wave equation in spherical coordinates can be expressed in terms of a velocity potential \(\phi(R, \tilde{t})\) as follows:

\[
c^2 \nabla^2 \phi(R, \tilde{t}) = \frac{\partial \phi(R, \tilde{t})}{\partial \tilde{t}},
\]

where \(P_n(x)\) is the \(n\)th Legendre polynomial and \(\phi_n\) is the component of \(\phi\) for \(n\)th Legendre polynomial. Upon substituting the second expression into the Laplace-transformed form of the first in above equation, the following ordinary differential equation is obtained:

\[
R^2 \frac{d^2 \phi_n}{dR^2} + 2R \frac{d\phi_n}{dR} - n(n+1) + R^2 \phi_n = 0,
\]

whose regular solution is given by [18]

\[
\tilde{\phi}_n(R, s) = B_n(s)\kappa_n(Rs),
\]

where \(s\) is the Laplace transform variable, an over-bar means Laplace transform, \(B_n(s)\) is the constant determined from the geometrical compatibility conditions and \(\kappa_n(Rs)\) is the \(n\)th order modified spherical Bessel function of the third kind. The pressure and particle velocity of acoustic fluid are related by

\[
p(R, \tilde{t}) = \phi(R, \tilde{t}), \quad u = -\frac{d\phi}{dR}(R, \tilde{t}).
\]

Using Eqs. (63) and (64), \(B_n(s)\) can be obtained and the desired Laplace-transformed analytical modal relation between the \(n\)th components of pressure and radial velocity on the external surface is expressed as [18]

\[
P_n(s) = -\kappa_n(s)/\kappa_n'(s)u_n(s).
\]

It should be noted that the above analytical modal pressure equation needs to be coupled with the equations of motion for the elastic sphere to bring about the coupling of the flexible structure with the surrounding external acoustic medium.

### 5.3. Modal equations of DAAs and the proposed model for a sphere

For spherical geometry, the DAA2 [4] and the proposed model (48) [11] can be expressed in terms of Legendre polynomials, which are summarized below:

\[
\text{DAA}_1: s + (1 + n)p_n = su_n, \quad (66)
\]

\[
\text{DAA}_2 \quad (1978): (s^2 + (1 + n)s + (1 + n)^2)p_n = s^2 + (1 + n)su_n, \quad (67)
\]

\[
\text{Proposed model: } (\lambda_n s^2 + (1 + \eta_n)s + (1 + n))p_n = \lambda_n s^2 + su_n. \quad (68)
\]

where \(\lambda_n\) is the \(n\)th mode weight parameter obtained from the parameterized matrix in Eq. (47) given by

\[
\lambda_n = 1/n \quad \text{for } n > 0 \quad \text{and} \quad \lambda_0 = 1 \quad \text{if } n = 0. \quad (69)
\]

The present modal form coincides with the spherical modal DAA2 [23] with a curvature correction term except \(n = 0\). As shown before, the corresponding matrix form of the DAA2 is difficult to implement and can suffer from a loss of computational stability.

Fig. 6 shows the poles of acoustic impedances for the exact solution (65), DAA2 (1978) (67) and proposed model (68) corresponding to the increasing order of the Legendre polynomial. In Fig. 6, the number of exact poles increases by one as the Legendre polynomial order increases. But the DAA2 and the proposed model have only two poles regardless of the modal order. For 0th and 1st modes, the proposed model captures the exact solution’s poles whereas the DAA2 (1978) does not. For the other modes, the magnitudes of real and imaginary values of the poles calculated by the DAA2 (1978) and the proposed model linearly increase. According to Fig. 6, the proposed model more accurately predicts each of the dominant analytic roots, especially imaginary part of these roots than the DAA2 (1978) in low modes (0–5) which most power is generally concentrated on. Therefore, the proposed model can more accurately predict the acoustic responses than DAA2, in general.

### 5.4. Modal acoustic–structure interaction equations for a sphere

For an elastic spherical shell, the radial and meridional displacements \((w, v)\) can also be expanded in term of the Legendre polynomials in the same way as the pressure and particle velocity have been expanded:

---

**Fig. 6.** Order-by-order specific acoustic impedance poles of exact solution, DAA2 and proposed model.
where \( w_n \) and \( \nu_n \) are components of \( w \) and \( \nu \) corresponding to the \( n \)th Legendre polynomial.

Using the preceding series expansions, the corresponding modal equations of motion for an elastic spherical shell with uniform thickness and isotropic material for each mode were given by Jungrer and Feit [27]. Combining the modal structure equation and acoustic models (65), (67) and (68), the modal acoustic-structure interaction equations for a spherical shell can be obtained in the following matrix form:

\[
\begin{bmatrix}
\lambda_n s^2 + A_n^{\text{sw}} & A_n^{\text{sw}} \nu_n \\
A_n^{\text{sw}} s^2 + A_n^{\text{sw}} & \mu \nu_n
\end{bmatrix}
\begin{bmatrix}
w_n \\
\nu_n
\end{bmatrix}
= \begin{bmatrix}
0 \\
-\mu p_n'
\end{bmatrix},
\]

(72)

where \( A_n^{\text{sw}} = \lambda_n(1 + \beta) \beta_n^{\text{sw}} \gamma_0 \),

\( A_n^{\text{sw}} = \lambda_n(1 + v + \beta_n^{\text{sw}}) \gamma_0 \),

\( A_n^{\text{sw}} = 2(1 + v) + \lambda_n \beta_n^{\text{sw}} \gamma_0 \),

and \( \mu = (\rho / \rho_i)(a/h) \), \( \gamma_0 = c_2^2 / c^2 \), \( \beta = (h/a)^2 / 12 \), \( \lambda_n = n(n + 1) \), \( \beta_n = (\lambda_n - 1 + v) \), and \( Q_n(s) \) and \( R_n(s) \) are given by

Exact: \( Q_n(s) = -\kappa(s) \), \( R_n(s) = \kappa(s) \).

Proposed: \( Q_n(s) = s(\lambda_n s + 1) \), \( R_n(s) = \lambda_n s^2 + (1 + \lambda_n) s + (n + 1) \).

DAA2: \( Q_n(s) = s(s + (n + 1)) \), \( R_n(s) = s^2 + (1 + n) s + (1 + n)^2 \).

An examination of the characteristic equation of Eq. (72) reveals that a structure interacting with acoustic medium has two types of poles: lightly damped structural poles and highly damped pressure poles. Among them, the most dominant poles are the lightly damped structural poles corresponding to the radial displacement. Fig. 7 shows the free-vibration root loci of Eq. (72) for \( n = 2 \). Material and geometric parameters of the same shown in Fig. 5. In particular, as the parameter \( \chi_r \) is varied, the roots-locus of the proposed model is shown, which indicates that the proposed model may be tailored to accurately capture the imaginary component of the exact poles (see Fig. 7).

Observe from the magnified root loci corresponding to \( n = 2 \) in Fig. 7 that \( \chi_r = 2 \) would faithfully reproduce the oscillatory of the exact analytical root. For other modes, similar root loci have been chosen for \( \chi_r \left( 2 \leq \chi_r \leq 7 \right) \). Table 1 presents the mode-by-mode dominant structural poles of the exact solution, the DAA2 (1978) and the proposed model. Note that the weighting parameter \( \chi_r \) of the proposed model has been chosen for each mode so that the dominant poles of the interaction equation using the proposed model can be stable and as close as possible to those of the exact solution. Notice the desirable \( \chi_n^3 \) increases almost one by one as the mode increases; hence, the formula adopted in Eq. (69). This is reflected in the construction of the discrete parameterization matrix, \( \mathbf{X} \) (see Eq. (47)).

5.5. Modal study of the proposed model for an infinite cylinder

Modal solution for wave equation exists in the cases of not only sphere, but also and infinite cylinder. But, the modal equation for an infinite cylinder is more complicated than for a sphere and the modal equation for the proposed model cannot be analytically derived. So, in this section, we will perform the modal study for an infinite cylinder using discrete long cylinder model like above cases of a sphere. The discrete model of a long cylinder will be explain in Section 6.2.1 in detail.

5.5.1. Modal equation of scattering pressure for an infinite cylinder

In the case of a cylinder, for the simplest case, we assume that the axial component of displacement vanishes and the circumferential and radial components \( \nu \) and \( w \), respectively, are independent of the axial coordinate \( z \). Therefore, the wave equation corresponding to cylindrical coordinates \( (r, \theta) \) shown in Fig. 8 is expressed as

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = p.
\]

(73)

The scattering pressure of the wave equation, Eq. (73), is derived as
Table 1
Mode-by-mode dominant structural roots for a spherical shell surrounded by water.

<table>
<thead>
<tr>
<th>Order</th>
<th>Optimal 1/2</th>
<th>Dominant structural roots</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Exact</td>
</tr>
<tr>
<td>2</td>
<td>1.83</td>
<td>-0.026 + 1.192i</td>
</tr>
<tr>
<td>3</td>
<td>2.92</td>
<td>-0.0069 + 1.505i</td>
</tr>
<tr>
<td>4</td>
<td>3.83</td>
<td>-0.0010 + 1.723i</td>
</tr>
<tr>
<td>5</td>
<td>4.91</td>
<td>-0.0001 + 1.889i</td>
</tr>
<tr>
<td>6</td>
<td>6.00</td>
<td>-0.0000 + 2.027i</td>
</tr>
<tr>
<td>7</td>
<td>6.93</td>
<td>-0.0000 + 2.148i</td>
</tr>
</tbody>
</table>

To predict the initial-value of the specific acoustic impedance, the values of \(\text{Fig. 12}\) for analytic equation, DAA₂ and the proposed model are plotted for increasing \(s\) as shown in Figs. 13–15. According to these figures, each \(\text{Fig. 12}\) converges to each constant value and especially, the exact and the proposed model converge to same value, 0.5 regardless of mode. In summary, the early-time

\[ p_s(r, \theta, \phi) = \sum_{n=0}^\infty p_n(r, \theta) \cos n\theta, \]  

(74)

where \(p_n\) is the component of \(p_s\) at the nth order. The Laplace-transformed scattered pressure satisfies

\[ \frac{\partial^2 p_n}{\partial r^2} + \frac{1}{r} \frac{\partial p_n}{\partial r} - \frac{n^2}{r^2} p_n = s^n p_n. \]  

(75)

The solution of Eq. (75) is recognized as the modified Bessel function of the second kind of order \(n, K_n(r)\). Applying the geometric compatibility conditions, Eq. (58), we obtain the relation between scattering pressure and velocity for nth order as

\[ K_n \frac{p_n}{K_n} = -\mathbf{u}_n. \]  

(76)

5.5.2. Specific acoustic impedance and early-time consistency for an infinite cylinder

Specific acoustic impedance which is the relation between pressure and particle velocity in fluid shows the characteristics of the proposed model. However, in the case of an infinite cylinder, both of the DAAs and the proposed approximation cannot be analytically expressed as modal equations like the case of a sphere. Therefore, it is not easy to compare the specific acoustic impedance of both approximations to analytic modal equation, (76). But, we already know that the scattering pressure and the particle velocity can be expressed in terms of \(\cos n\theta, Eq. (74)\). Therefore, prior- and post multiplying the eigenvector expressed as the form of \(\cos n\theta\) to the boundary element matrices of both approximations, the modal equations can be derived. To this end, we develop the discrete boundary models of the DAA₂ and the proposed model, (48) corresponding to the three dimensional long cylinder as shown in Fig. 34. From the discrete models, the modal equations of DAA₂ and the proposed model are obtained.

Now, using the analytic modal relation (76) and modal equations of both approximations, the specific acoustic impedances are drawn by the ratio, the characteristic structural wavelength \((\lambda_s)\) for the surface motion to acoustic wavelength as shown in Figs. 9–12. In this case, to compensate the discrete error, the ideal geometric value (0.25) of \(\left[ \int \mathbf{u}_n(dS) \int dS \right]^{-1}\) of the proposed model at each node is used. As shown in these figures, at \(n = 0, 1\) the modal specific acoustic impedances of the proposed model are almost same as exact, but DAA₂ introduces large error in transient regions. But, in high mode as shown in Fig. 12, both of the approximations cannot accurately approach the exactness in transient regions like the case of a sphere. In conclusion, the proposed model has an advantage not in high modes but in low modes.

Now, the early-time consistency is checked in the case of a cylinder at each mode using the initial-value theorem:

\[ \lim_{t \rightarrow 0} \left[ \frac{p_n(t)}{\mathbf{u}_n(t)} \right] = \lim_{s \rightarrow \infty} \left[ \frac{p_n(s)}{\mathbf{u}_n(s)} \right] \]  

(77)
consistency of the proposed model still maintains in the case of a cylinder as

$$\lim_{t \to 0} \frac{\rho_h(t)}{u_n(t)} = \begin{cases} \delta(0) - 0.5 & \text{for the exact solution and the,} \\ \delta(0) & \text{proposed model} \end{cases}$$

(78)
6. Numerical simulation

For evaluation of the proposed model, we present the transient responses of submerged spherical and cylindrical shells using the proposed pressure model and compare them with analytical solution and various approximations’ results. In particular, these examples are solved by using discrete proposed model (48) to validate the implementation of the proposed model for complex geometric problems.

6.1. A submerged spherical shell excited by impulsive pressure

The submerged spherical shell excited by a plane step wave in Fig. 16 has been analyzed in many papers [19,25]. For this example problem, many of the existing approximate models [4,28] perform rather well as shown in Figs. 17 and 18, with the exception of the DAA1.

In this case, the transient responses rapidly reach steady state since a constant pressure level surrounds the spherical shell after incident wave passes through the sphere. For the present evaluation, we have chosen the submerged spherical shell excited by cosine-type impulsive pressure as shown in Fig. 5 and a total of thirteen modes are utilized in the series solutions. As can be observed, the solutions are characterized by the velocity and pressure transients during the early time period to the steady state oscillations during the late-time period. This means that a large number of modes participate with different weights at different time window, thus directly exposing the roles of different characteristic poles as shown in Fig. 6.

6.1.1. Results using modal equations

Figs. 19 and 20 show the radial velocity responses on the spherical surface of Fig. 5 at $\theta = 180^\circ$ and $0^\circ$ using the mode summation of the coupled modal Eq. (72) of the proposed model. The $n = 0–5$ modes fully describes early-time responses and the late responses also converge as the number of participated modes is increased.

Figs. 21 and 23 show analytic responses and the results calculated by using modal forms of the proposed model (68), DAA1 (66) and DAA2 (1978) (67). The analytic responses are calculated by using Huang’s analytic equation [19]. As already mentioned, in particular, the responses show that the radial velocity is rapidly
changing at early time, and followed by the periodic oscillatory responses. Observe that for the late-time period, the DAA2 (1978) and the proposed model follow with a reasonable phase and amplitude fidelity of the Huang’s radial velocity responses, but DAA1 shows large error in entire time range (see Figs. 22 and 24). The DAA2 (1978), however, over(under)-estimates the early-time peak at $\phi = 180^\circ$ while the proposed model predicts the early-time responses with high accuracy. This difference is caused by the inaccuracy of the low-mode impedance poles of the DAA2 (1978) shown in Fig. 6 and early-time inconsistency.

The early-time inconsistency manifests itself in impulse responses function of the impedance. To this end, the cosine-type impulse velocities such as the cosine-type pressure to Eq. (72) were applied to initiate the pressure response. Figs. 25 and 26 show the impulse pressure response and its error compared to the analytic result at $h = 180^\circ/C_14$. Note that the two DAAs have large error at initial time due to the early-time inconsistency, whereas the proposed model closely follow the analytical initial response since the present model is consistent with respect to the analytic initial responses. The early-time consistency is an important characteristic in pressure source identification or an array of acoustic inverse problems.

Now, returning back in Fig. 5, that is, the frequency responses of a spherical shell subjected to cosine-type impulse pressure is reexamined. Figs. 27 and 28 show the frequency responses on the surface at $\theta = 180^\circ$ and $0^\circ$. In these frequency responses, we introduced structural loss factor [27], $\eta$ by multiplying $c_i/c$ by $(1 + i\eta/2)$. Observe the DAA2 (1978) captures more accurately the low-frequency peaks of structural modes than the proposed model; however, the proposed model accurately estimates intermediate- and high-frequency peaks than the DAA2.

6.1.2. Responses using discrete boundary element equations

The discrete boundary element models corresponding to DAA2 (1978) and proposed model, Eq. (48) were constructed. First, in order to validate the fidelity of the present discrete equation, the number of the boundary elements were increased from 385 to 865. Fig. 29 shows the convergence of specific acoustic impedance poles of discrete proposed model toward the those of the modal corresponding theoretical model equation. As the number of element increases, the specific acoustic impedance poles of the proposed model using parameterized discrete matrix (47) approach

![Fig. 21. Radial velocity at $\theta = 180^\circ$ using modal solution ($n = 0$–12).](image1)

![Fig. 22. Error of radial velocity with respect to exact solution on a submerged spherical shell at $\theta = 180^\circ$ using modal solution ($n = 0$–12).](image2)

![Fig. 23. Radial velocity at $\theta = 0^\circ$ using modal solution ($n = 0$–12).](image3)

![Fig. 24. Error of radial velocity with respect to exact solution on a submerged spherical shell at $\theta = 0^\circ$ using modal solution ($n = 0$–12).](image4)
the specific acoustic impedance poles of modal equation (68) with the weighting parameter, \( \gamma_n \) (69).

Now, in order to calculate transient responses, finite and boundary discrete element models in interaction Eq. (59) are constructed as 600 quad4 elements for both structure and acoustic equation and the coupled interaction Eq. (59) are solved by using the staggered solution [30] at each time step. In order to avoid discontinuity, we apply an impulsive pressure expressed in the form of Gaussian function as

\[
p_I(t, \theta) = \cos \theta \frac{5}{\sqrt{\pi}} \exp[-5(t - 0.5)^2]H(-\cos \theta),
\]

(79)

\[
H(x) = \begin{cases} 
1 & \text{if } x > 0, 
1/2 & \text{if } x = 0, 
0 & \text{if } x < 0.
\end{cases}
\]

(80)

'Radial velocities using modal equation and the discrete models of the proposed model at \( \theta = 180^\circ \) and \( 0^\circ \) are shown in Figs. 30 and 31. The results calculated by the discrete models almost approach the results by modal equation. Figs. 32 and 33 show the radial velocities estimated by discrete models of various approximations.

We have included the result obtained by the fluid finite element provided by ANSYS commercial program [29] for which the absorbing boundary [31] is used for this example problem. In modeling this problem by ANSYS code, the exterior acoustic domain up to \( 2R \) (two times of radius of a sphere) from the origin of a sphere is modeled by the fluid volume elements. The elastic sphere and the fluid volume are treated as if they are a structure, and the absorbing boundary is modeled with 600 4-noded boundary elements. The responses calculated by the finite fluid elements and absorbing boundary show good early-time agreement and have a similar trend compared to the analytic results. But, the error between the calculated results and analytic results increases in late-time responses compared to those of the proposed model.

6.2. Application to an infinite cylindrical shell

The scattering pressure by a cylinder had been dealt with by many previous researchers. Since Carrier [26] first derived the modal solution for an infinite submerged cylindrical shell interacting with exterior fluid, Mindlin and Bleich [1], Junger [32], Hayward [33] and other researchers have investigated the problem.
of an elastic infinite cylindrical shell excited by incident waves in fluid. Nevertheless, obtaining a rigorous solution remains a challenge because the analytic inverse Laplace transform of a cylindrical wave equation does not exist. Mindlin and Bleich [11] first proposed the early-time approximation and Haywood [33] presented the high-order approximation for solving the relation between scattering pressure and velocity of a cylindrical wave. Huang [20] obtained modal solutions using contour integrate technique and convolution integration to perform the inverse Laplace transform of a cylindrical wave. Geers [34] also investigated the modal solution for an infinite cylindrical shell submerged in fluid using residual potentials, but it is very complex.

In our correlation studies, we will use Huang's results [20] as accurately traced as possible while recomputing the DAA's results. In the modeling of an infinite long cylinder, we use a sufficiently long cylindrical shell and to consider it to be an infinite cylinder.

6.2.1. A submerged infinite cylindrical shell subjected to plane waves

The transient responses of an elastic cylindrical shell subjected to a step plane wave have been obtained by Huang [20] and other researchers [34]. Huang obtained his analytic result within limit of series solution. In numerical methods, transient responses of a submerged cylinder can be calculated by staggering solving interaction equations, Eq. (59) in time sequence. We approximate a long cylindrical shell enough to consider it as an infinite cylindrical shell with the length (L) to the radius (a) ratio to be 40. The cylinder has the following geometric and material parameters: \( v = 0.3, L/a = 40, h_c/a = 0.029 \) and \( \rho_c/\rho = 7.8 \) as shown in Fig. 34, and modeled the long cylindrical shell and surrounding acoustic fluid by finite element and boundary element models, respectively. We first examine a long cylindrical shell subjected to incident step plane wave such as adopted in Huang's results [20]. The incident plane wave's pressure, \( p' \) and velocity, \( u' \) are defined in [34].

Fig. 35 shows the radial velocities on surface of a submerged cylindrical shell at \( \theta = 180^\circ \) and 0°. Here, the velocity response at \( \theta = 0^\circ \) is exactly zero until \( tc/a = 1 \) because it takes \( tc/a \approx 1 \) for the excitation by incident wave to arrive at \( \theta = 0^\circ \) through the cylindrical shell. But, since Huang's result is obtained by finite mode summation, the velocity response at \( \theta = 0^\circ \) is not exactly zero for \( 0 \leq tc/a \leq 1 \). The results calculated by the proposed model and DAA's accurately approach Huang's at early time with high accuracy and also converge to Huang's results at late time. At \( \theta = 0^\circ \) both of proposed model and DAA's show error at \( tc/a = 2 \), because of the discontinuity of incident pressure at \( tc/a = 2 \). Finally, in a cylindrical shell subjected to incident plane wave, the proposed model has a similar performance to DAA's because the responses rapidly go to steady state at early time. Therefore, all of approximations show good accuracy.

6.2.2. A submerged infinite cylindrical shell subjected to cosine-type impulse pressure

In order to examine the transient dynamic responses of an infinite under-water cylindrical shell, we perform numerical simulation for same long cylinder subjected to cosine-type impulse pressure. The cosine-type impulse pressure acting on the surface of a cylindrical shell is defined in the form of Gaussian function (79). The discrete pressure equation and the elastic structural equations are the same ones generated in the previous section.

![Fig. 29. Convergence of specific acoustic impedance poles in discrete proposed model.](image)

![Fig. 30. Radial velocity on a submerged spherical shell at \( \theta = 180^\circ \) using modal equation and discrete model of proposed model.](image)
In order to evaluate the proposed model, we compared the result of the proposed discrete boundary model with that of the DAAs' and the domain based approximate model (Ansys) [29]'s results. Fig. 36 shows the radial velocities on the surface of the cylindrical shell at $\theta = 180^\circ$ and $0^\circ$. Both the proposed model and the DAA2 show similar responses and the results of both models are close to the results obtained by using the 2D ANSYS model.

We observe that the problem of a cylinder subjected to incident pressure waves is essentially a two-dimensional problem. For this particular two-dimensional problem, little difference in responses are noticed. While not comprehensive, we are tempted to conclude that various second-order models would yield adequate results, even perhaps the DAA1.

6.3. Shock analysis of a complex stiffened cylinder

So far the proposed approximate model is applied to simple structures, a submerged sphere and an infinite cylinder. For such problems, the proposed model has been shown to yield accurate coupled interaction responses. The fidelity of the present model will now be assessed by analyzing nontrivial problems, viz., shock analysis of complex stiffened cylinder subjected to a external charge. To this end, a steel ring-stiffened, circular cylinder of finite length with flat aluminum end caps in [21] is analyzed as the this structure was a testbed of Defense Nuclear Agency (DNA) and the
Office of Naval Research (ONR) of USA for validating the DAA-based computer code for predicting underwater explosion [21].

6.3.1. Computational model generation

The structure model is a ring-stiffened finite cylindrical shell containing the internal structure and both flat ends consist of aluminum caps. Dimensions of the model are taken from [35] and the material properties of steel and aluminum are chosen as \( E = 2.9 e 7 \) psi, \( \rho = 0.2836 \) lb/in\(^3\), \( v = 0.3 \) and \( E = 9.858 e 6 \) psi, \( \rho = 0.0975 \) lb/in\(^3\), \( v = 0.33 \). The finite element model of the steel ring-stiffened, right circular cylinder consists of 5016 elements and 17,616 degrees of freedom.

According to [21], in the experiment of underwater explosion, the side-on loading was produced by the underwater explosion of a 201.5 pound tapered charge of HBX-1 exploded at a standoff distance of 69 feet. Both the model and the charge were submerged to a depth of 35 feet. The typical pressure–time history from a pressure transducer located at about the stand-off distance is reproduced from [21]. Fig. 37 shows the measured incident pressure from the pressure transducer and its simplified incident pressure for simulation.

6.3.2. Submerged shock analysis

The complex structure is surrounded by water whose density and sound speed are \( \rho = 0.0365 \) lb/in\(^3\), \( c = 60000 \) in/s, respectively. Therefore, exterior acoustic field strongly interacts with the structural surface of the stiffened cylinder. As can be seen from Eq. (59), the structural finite element model and the exterior fluid boundary element model are coupled with each other and shock responses by the incident wave shown in Fig. 37 can be also simulated. In reference paper, [21], each measured point is located as shown in Fig. 38.
The present paper employs a weighted combination of the retarded and advanced potentials to derive a stable second-order approximate external acoustic model with a weighting parameter, $\chi$, for the case of continuum approximate model equation, and a corresponding weighting matrix $X$ for the discretized form of the model equation that is applicable for general interaction surfaces.

The parametrization of the weighting parameter $(\chi)$ and the discrete counter part $(X)$ has been achieved by comparing the modal equation of the present model vs. the pole-zero characteristics of the analytical solution of an elastic sphere interacting with external transient acoustic pressure fields. The present discrete pressure model (48) with the weighting matrix $X$ may be viewed as a stabilized DAA$\gamma$ with a curvature correction, with an added desirable early-time consistency for correctly capturing impulse responses that may prove important for inverse problems.

The performance of the proposed model have been assessed by applying the present external acoustic model equation to vibration and transient response analysis of several example problems. To this end, the analytical series solutions have been generated for structures subjected to plain impulse incident acoustic pressure, cosine-type impulse incident pressure as well as plane step incident wave. The same problems are solved by the present discrete model pressure equation and the DAA$\gamma$ equation. Comparisons show that the present model correctly capture initial impulse responses whereas the DAA$\alpha$ and DAA$\beta$ do not correctly capture the initial impulse responses.

In order to further validate the present discrete model Eq. (48), eigenvalues of the coupled discrete acoustic–structural interaction equations employing the present discrete acoustic model and a standard finite element structural analyzer are compared with those of the corresponding coupled modal equations for the spherical problem. Results show that, as the number of the discrete elements increases, the coupled discrete eigenvalues approach to those of the modal eigenvalues. A closer examination of the numerical results conducted so far show that the DAA$\alpha$ does trace somewhat more accurately the late-time steady state responses than the present parameterized model equation when the coupled steady state responses are dominated by structural rather than acoustic contributions. On the other hand, the present parameterized pressure model equation exhibits substantially enhanced accuracy for early-time responses that are desirable as a potential candidate model for system identification involving acoustic–structure interactions.

The present model has been assessed through a correlation study of a shock response experiment of a complex stiffened circular cylinder. The simulation results show experimental data reasonably faithfully both in terms of their pressure peaks and their phases. Hence, the present discrete pressure model is recommended for use not only in transient simulations for arbitrary geometries but also as a model equation for interaction system identification.

Nevertheless, we believe there is room for further improvements in capturing the highly oscillatory components in very flexible structures like balloons. This would call for a deeper understanding of the plane wave approximation used in the construction of the present model so that an improved correction of the present plane wave approximation may be needed. Such avenues are being actively studied.

### Acknowledgement

This research was supported by National Research Laboratory program through the Korea Science and Engineering Foundation funded by the Ministry of Education, Science and Technology (ROA-2005-000-10112-0) and BK21. It is a pleasure to acknowledge the visiting appointment of K. C. Park that enabled the collaborative work reported herein.
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